Problem Set 9

(1) Reading: Section 6.4 from the beginning of 6.4.2 (p 347) to the end. Also Section 6.5 up to the end of the proof of Theorem 2 (p 360).

For each of the next four questions, you should assume U is an open bounded subset of $\mathbb{R}^n$, and L is an elliptic operator in nondivergence form as given in equation (2) of Section 6.4 of Evans (page 344).

(2) Suppose U has the interior ball property and a is a smooth function on ∂U with $a(x) > 0$. Moreover suppose the function c in the definition of L has $c(x) \geq 0$. Show that for any given $f \in C(\overline{U}), g \in C(\partial U)$, the Robin problem

$$Lu = f \text{ in } U$$
$$\partial_\nu u + a(x)u = g \text{ on } \partial U$$

has at most one solution $u \in C^2(U) \cap C^1(\overline{U})$.

(3) Suppose $c \geq 0$ and $u \in C^2(U) \cap C(\overline{U})$ satisfies the inequality $Lu \leq 0$ in U, and $u \leq 0$ on ∂U. Show that either $u < 0$ in U or $u \equiv 0$ on U. Hint: use the strong maximum principle.

(4) Suppose $u \in C^2(U) \cap C(\overline{U})$ satisfies the inequality $Lu \leq 0$ in U, and $u \leq 0$ in U. Show that either $u < 0$ in U or $u \equiv 0$ on U. (Hint: you can't use the strong maximum principle directly because you don't know the sign of c. But you can write $c = c^+ + c^-$ as the sum of a nonnegative and nonpositive function. Show that if you replace c with c^+ in the definition of L, you still get $Lu \leq 0$.)

(5) Suppose there exists $w \in C^2(U) \cap C^1(\overline{U})$ with $w > 0$ on $\overline{U}$ and $Lw \leq 0$ in U. Show that if $u \in C^2(U) \cap C(\overline{U})$ satisfies $Lu \leq 0$ in Ω, then u/w can't have a nonnegative maximum in U unless u is a constant multiple of w.

(6) Suppose $U \subset \mathbb{R}^n$ is open and bounded, and $u \in H^1(U)$ is a weak solution to the divergence form equation

$$\nabla \cdot (A \nabla u) = 0$$

in U, where A is positive definite and smooth on U. Show that there exists some constant C such that

$$\int_U \eta^2 |\nabla u|^2 \, dx \leq C \int_U |\nabla \eta|^2 u^2 \, dx$$

for any $\eta \in C_0^\infty(U)$. (I asked a version of this question in a problem set from last semester too!)