Math 633 Sample Midterm

Do at least **three** of the following five questions. If you complete more than three questions, your grade will be based on the best three.

(1) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Consider the boundary value problem

$$\begin{aligned} -\triangle u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

(a) State what it means for $u \in H_0^1(\Omega)$ to be a weak solution of the boundary value problem.

(b) Show that if $f \in L^{\frac{2n}{n+2}}(\Omega)$, then there exists a weak solution $u \in H_0^1(\Omega)$ to the boundary value problem above.

(2) Suppose $u \in W^{1,p}(\mathbb{R})$ for $1 < p < \infty$. Prove that u is Hölder continuous with exponent $1 - 1/p$, and

$$|u(x) - u(y)| \leq |x - y|^{1-1/p} \|u'\|_{L^p(\mathbb{R})}$$

for all $x, y \in \mathbb{R}$. Here u' indicates the weak derivative.

(3) (a) Give the definition of the Sobolev spaces $W^{1,p}(B(0,1))$ and $W_0^{1,p}(B(0,1))$.

(b) Let $u \in W^{1,2}(B(0,1))$ and suppose that

$$\limsup_{x \to y} u(x) \leq 0 \text{ for all } y \in \partial B(0,1)$$

Show that $u^+ = \max(u, 0) \in W_0^{1,2}(B(0,1))$.

(4) Suppose that $\Omega \in \mathbb{R}^n$ is a bounded domain, and $A_{ij} \in C^\infty(\Omega)$ for each $1 \leq i, j \leq n$. Let

$$Lu = -\nabla \cdot (A \nabla u)$$

where A is symmetric and positive definite. Suppose that $f \in L^2(\Omega)$ and u is a weak solution to

$$\begin{aligned} Lu &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Show that there exists some constants $C_1, C_2 > 0$ such that

$$C_1 \int_\Omega |u|^2 dx \leq \int_\Omega |\nabla u|^2 dx \leq C_2 \int_\Omega (f^2 + u^2) dx.$$

(5) Suppose that $u \in C^1(\mathbb{R}^n_+) \cap W^{1,p}(\mathbb{R}^n_+)$. Show that

$$\|u\|_{L^p(\partial \mathbb{R}^n_+)} \leq C \|u\|_{W^{1,p}(\mathbb{R}^n_+)}$$

for some $C \in \mathbb{R}^n$. Explain why this implies that the map $T : u \to u|_{\partial \mathbb{R}^n_+}$ extends to a map $T : W^{1,p}(\mathbb{R}^n_+) \to L^p(\partial \mathbb{R}^n_+)$.