1. Review

The final exam in this class will be on Monday May 1 from 8-10am and will cover all material from this class, albeit with an emphasis on material covered since the last midterm. Roughly speaking, the material in this class is divided into three major sections, of unequal weight.

1.1. **Measure and Integration.**

This is basically the midterm material: Lebesgue measure, the Lebesgue integral, and the convergence theorems, together with Fubini's theorem. Topics in this section include outer measure, measurable sets, Lebesgue measure on $\mathbb{R}^n$; measurable fuctions, Lebesgue integrals; dominated convergence, monotone convergence, Fatou's lemma; and Fubini's theorem. See also the midterm review sheet.

1.2. **Function spaces as metric spaces.**

The second major topic is our introduction to sets of functions as metric spaces. You should be familiar not only with the basic properties of metric spaces, but also with $C(\mathbb{R}^n)$, $L^1(\mathbb{R}^n)$, $L^p(\mathbb{R}^n)$, and their basic properties. The ℓ^p spaces are not so much a part of this course per se, but are a good source of examples, and they may make an appearance on the exam.

1.3. **Differentiability of the integral.**

Here the main topics are the Lebesgue Differentiation Theorem and the absolutely continuous functions. The differentiation theorem in particular serves as an excellent introduction to a number of important ideas from harmonic analysis: covering lemmas, maximal functions, etc. and you should be familiar with the main ideas in the proof. The same goes for the main theorem of absolute continuity, namely that absolutely continuous functions are the ones that satisfy the second fundamental theorem of calculus.

Below I've listed twelve problems which are roughly representative of the material we've covered and the kind of problem I expect you to be able to do. Two of these (or parts of two of these) will make an appearance on the exam in whole or in part. Nevertheless, *you should not limit your study to these problems only.*

Problem 1. *Show that a set $A \subset \mathbb{R}^n$ is measurable if and only if for every $\varepsilon > 0$ there exists a compact set K such that $K \subseteq A$ and*

$$m^*(A \setminus K) \leq \varepsilon.$$

Problem 2. *Suppose $A \subset \mathbb{R}^n$ has finite measure, and $f : A \to \mathbb{R}$ is a bounded function. Show that there exists a sequence $\{g_k\}$ of ISFs such that $g_k \to f$ pointwise. Do the same when $f : \mathbb{R}^n \to \mathbb{R}$ is a general integrable function.*

Problem 3. *Prove the Dominated Convergence Theorem in the case where the dominating function g is an ISF. Use this result to prove the Dominated Convergence Theorem in general.*

Problem 4. *Suppose* $f \in L^1(\mathbb{R})$ *and* $g \in C^1(\mathbb{R})$*. Show that*

$$f * g(x) = \int f(y)g(x-y)dy$$

is differentiable (at every $x \in \mathbb{R}$*).*

Problem 5. *Suppose* $f : \mathbb{R}^2 \to \mathbb{R}$ *is nonnegative and*

$$\int \left(\int f(x,y)dx \right) dy = 1.$$

Show that f *is integrable. By means of a counterexample, show that the nonnegativity condition here is necessary.*

Problem 6. *Suppose* (X, d) *is a metric space, and* $\{a_n\}$ *and* $\{b_n\}$ *are Cauchy. Show that the sequence* $\{d(a_n, b_n)\} \subset \mathbb{R}$ *converges.*

Problem 7. *Show that* $C(\mathbb{R})$ *is complete.*

Problem 8. *Show that* $L^1(\mathbb{R})$ *is complete.*

Problem 9. *Suppose* $f \in L^1(\mathbb{R}^n)$*, and define* $f_h : \mathbb{R}^n \to \mathbb{R}$ *by* $f_h(x) = f(x+h)$*. Show that*

$$\lim_{h \to 0} \|f_h - f\|_{L^1(\mathbb{R}^n)} = 0.$$

By means of a counterexample, show that this does not hold if L^1 *is replaced by* L^∞*.*

Problem 10. *Suppose* $f \in L^1(\mathbb{R}^n)$*. Show that*

$$m(\{x \in \mathbb{R}^n | Mf(x) > \alpha\}) < \frac{3^n}{\alpha} \|f\|_{L^1(\mathbb{R}^n)}$$

for all $\alpha > 0$*. Here* Mf *is the Hardy-Littlewood maximal function.*

Problem 11. *Use the previous problem to prove the Lebesgue differentiation theorem.*

Problem 12. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be integrable. Show that for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* $E \subset \mathbb{R}$ *has* $m(E) < \delta$*, then*

$$\int_E |f| < \varepsilon.$$

Use this to prove that

$$F(x) = \int_a^x f(t)dt$$

is absolutely continuous.