1. Midterm Review

The midterm in this class will be on Friday March 3 from 1-3pm and will cover roughly the material covered in class up to the end of class on February 24. Roughly speaking this material is divided up into three major themes.

1.1. **Lebesgue measure.**

You should be familiar with the definition of *outer measure*, *measurable sets*, and *Lebesgue measure* both on $\mathbb{R}$ and $\mathbb{R}^n$. Be familiar with the basic properties of each of these objects, and be prepared to prove the basic properties using the definitions.

1.2. **Lebesgue integral.**

You should be familiar with the definition of a *measurable function*, the basic properties of measurable functions, and their proofs. In addition, you should be comfortable with the definition of the *Lebesgue integral* in terms of integrable simple functions, and know the basic properties of the Lebesgue integral and their proofs.

1.3. **Convergence theorems.**

You should know the major convergence theorems (dominated convergence, monotone convergence, and Fatou) and be familiar with their proofs. In addition, you should be able to use the convergence theorems to prove additional facts, such as the equality of the Riemann and Lebesgue integrals.

Below I've listed ten problems which are roughly representative of the material we've covered and the kind of problem I expect you to be able to do. Two of these (or parts of two of these) will make an appearance on the exam in whole or in part. Nevertheless, *you should not limit your study to these problems only.*

Problem 1. *Show that the outer measure of an open interval $(a, b) \subset \mathbb{R}$ is equal to $b - a$.*

Problem 2. *Show that a set $A \subset \mathbb{R}^n$ is measurable if and only if for every $\varepsilon > 0$ there exists an open set U such that $A \subseteq U$ and*

$$m^*(U \setminus A) \leq \varepsilon.$$

Problem 3. *Show that if $A_k \subset \mathbb{R}^n$ are measurable and disjoint, then*

$$m(\cup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} m(A_k).$$

Problem 4. *Suppose $f_k : \mathbb{R} \to \mathbb{R}$ are measurable functions, and $f(x) = \sup_k f_k(x)$ is finite for each $x \in \mathbb{R}^n$. Show that f is measurable.*

Problem 5. *Suppose $A \subset \mathbb{R}^n$ has finite measure, and $f : A \to \mathbb{R}$ is a bounded function. Show that there exists a sequence $\{g_k\}$ of ISFs such that $g_k \to f$ pointwise. Do the same when $f : \mathbb{R}^n \to \mathbb{R}$ is a general integrable function.*

Problem 6. *Suppose f and g are nonnegative integrable functions. Show that*

$$\int f + \int g = \int (f + g).$$

Problem 7. *Prove Egorov's Theorem: suppose* $\{f_k\}$ *is a sequence of measurable functions defined on a measurable set* E *of finite measure, and* $f_k \to f$ *pointwise on* E. *Then for each* $\varepsilon > 0$ *there exists a subset* $A \subset E$ *with* $m(E \setminus A) < \varepsilon$ *and* $f_k \to f$ *uniformly on* A.

Problem 8. *Prove the Dominated Convergence Theorem in the case where the dominating function* g *is an ISF. Use this result to prove the Dominated Convergence Theorem in general.*

Problem 9. *Suppose* f *is an integrable function and* g *is continuous, bounded and measurable. Show that*

$$f * g(x) = \int f(y)g(x-y)dy$$

is continuous.

Problem 10. *Prove Lusin's Theorem: Suppose* E *is a set of finite measure and* $f : E \to \mathbb{R}$ *is measurable. Then for every* $\varepsilon > 0$, *there exists a subset* $F \subset E$ *with* $m(E \setminus F) < \varepsilon$, *and* f *is continuous on* F. *Hint: use Egorov's theorem.*