1. Problem Set 10

Problem 1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be the characteristic function of the unit ball centered at the origin in $\mathbb{R}^n$. Show that there exists a constant c such that*

$$\frac{c}{\alpha} \leq m(\{x|Mf(x) > \alpha\})$$

for all sufficiently small $\alpha > 0$. (Note that by the Hardy-Littlewood theorem, it follows that

$$\frac{c}{\alpha} \leq m(\{x|Mf(x) > \alpha\}) \leq \frac{3^n}{\alpha}$$

for all sufficiently small α.)

Problem 2. *Suppose f is integrable on $\mathbb{R}^n$. Show that $Mf \in L^1(\mathbb{R}^n)$ if and only if f is identically zero. (Hint: show that*

$$Mf(x) \geq \frac{C}{|x|^n + 1}$$

for some constant C.)

Problem 3. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, and let $Q_\alpha(x)$ be the n-cube of side length α in $\mathbb{R}^n$ centred at x. Show that*

$$\lim_{\alpha \to 0} \frac{1}{\alpha^n} \int_{Q_\alpha(x)} f(y)dy = f(x)$$

for almost every $x \in \mathbb{R}^n$.

Problem 4. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies the inequalities*

$$\chi_{[-1,1]} \leq \Phi \leq 2\chi_{-2,2}.$$

and let $\Phi_t(x) = t^{-1}\Phi(x/t)$ for $t > 0$. Show that there exists a constant C depending only on Φ such that for any $f \in L^1(\mathbb{R})$,

$$\lim_{t \to 0+} f * \Phi_t(x) = Cf(x)$$

for almost every $x \in \mathbb{R}$.

Problem 5. *Suppose E is a measurable set and*

$$m(E \cap B) \geq \frac{1}{2} m(B)$$

for all balls B. Show that $m(E^c) = 0$. (Hint: Apply the Lebesgue differentiation theorem to a well-chosen characteristic function.)

Problem 6. *Let $x \in \mathbb{R}^2$, $\alpha > 0$. Fix coordinates (x_1, x_2) in $\mathbb{R}^2$ and define $R_\alpha(x)$ to be the rectangle*

$$R_\alpha = (x - \sqrt{\alpha}, x + \sqrt{\alpha}) \times (x - \alpha, x + \alpha)$$

Find an integrable function f such that

$$\lim_{\alpha \to 0} \frac{1}{m(R_\alpha(x))} \int_{R_\alpha(x)} f(y)dy \neq f(x)$$

for all x in a set E of positive measure, or show that no such f exists.

Problem 7. *Give an example of a function which is absolutely continuous but not Lipschitz.*