1. Problem Set 2

Problem 1. *Show that replacing open intervals with closed intervals in the definition of outer measure yields an equivalent definition.*

Problem 2. *Show that A is measureable if and only if for every $\varepsilon > 0$ there exists an open set U such that $A \subseteq U$ and $m^*(U \setminus A) < \varepsilon$.*

Problem 3. *Show that A is measureable if and only if for every $\varepsilon > 0$ there exists a closed set E such that $E \subseteq A$ and $m^*(A \setminus E) < \varepsilon$.*

Problem 4. *Suppose $\{A_k\}, \{B_k\}$ are countable collections of measureable sets with finite measure. Suppose moreover that $A_1 \subset A_2 \subset \dots$ and $B_1 \supset B_2 \supset \dots$. Show that*

$$m\left(\cup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k)$$

and

$$m\left(\cap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_k).$$

Problem 5. *Suppose that $\{A_k\}$ is a countable collection of sets. Define*

$$\limsup_{k\to\infty} A_k = \{x | x \in A_n \text{ for infinitely many } n \in \mathbb{N}\}$$

and

$$\liminf_{k\to\infty} A_k = \{x | x \in A_n \text{ for all but finitely many } n \in \mathbb{N}\}.$$

Find an example to show that

$$\liminf_{k\to\infty} A_k \neq \limsup_{k\to\infty} A_k$$

in general.

Problem 6. *Suppose that $\{A_k\}$ is a countable collection of measureable sets. Show that $\liminf_{k\to\infty} A_k$ and $\limsup_{k\to\infty} A_k$ are measureable.*

Problem 7. *Suppose that $\{A_k\}$ is a countable collection of measureable sets, with the property that*

$$\sum_{n=1}^{\infty} m(A_k) < \infty.$$

Show that $m\left(\limsup_{k\to\infty} A_k\right) = 0$.

Problem 8. *Let $A \subset \mathbb{R}$ be a set with the property that $m^*(A) > 0$. Show that for each $0 < \alpha < 1$ there exists an open interval I such that*

$$m^*(A \cap I) \geq \alpha m^*(I).$$

(You can think of this as saying that A must contain "most of an interval".)