1. Problem Set 3

Problem 1. *Show that if f is continuous and g is measureable, then $f \circ g$ is measureable. Give a counterexample to show that $g \circ f$ may not be measureable.*

Problem 2. *Show that if f and g are measureable, then fg is measureable. (Hint: first show this for $f = g$, then write a general fg in terms of squares.)*

Problem 3. *Suppose A is a measureable set of finite measure, and $\{f_n\}$ are a sequence of measureable functions that converge pointwise to f on A. Show that for any $\varepsilon > 0$ there exists a measureable $B \subset A$ so that $m(A \setminus B) < \varepsilon$ and $f_n \to f$ uniformly on B.*

The next five problems deal with the following function. Suppose $x \in [0,1]$. Define $\varepsilon_1(x) = 1$ if $x \geq 1/2$ and 0 otherwise, and define $\varepsilon_n(x)$ recursively by

$$\varepsilon_n(x) = \begin{cases} 1 & \text{if } \sum_{k=1}^{n-1} \frac{\varepsilon_k}{2^k} + 2^{-n} \leq x \\ 0 & \text{otherwise} \end{cases}$$

Define

$$g_n(x) = \sum_{k=1}^{n} \frac{2\varepsilon_k(x)}{3^k}.$$

and

$$g(x) = \lim_{n \to \infty} g_n(x).$$

Problem 4. *Show that g is well defined and surjective as a map $g : [0,1] \to C'$, where C' is a subset of the Cantor set C from Problem Set 1 such that $C \setminus C'$ is countable. Show that g is strictly increasing, and hence one to one. (Note that this shows the Cantor set is uncountable.)*

Problem 5. *Show that g is continuous on a set of measure 1.*

Problem 6. *Show that if A is measureable, then each $\varepsilon_n^{-1}(A)$ is measureable but $g^{-1}(A)$ may not be measureable. This is (one reason) why we don't use this definition of a measureable function!*

Problem 7. *Note that $\inf\{C' \cap [x,1]\}$ lies in C'. With this in mind, define a new function $c : [0,1] \to [0,1]$ by*

$$c(x) = g^{-1}(\inf(C' \cap [x,1])).$$

(This is sometimes called the Cantor function, or the Cantor-Lebesgue function.) Show that c is continuous and surjective. Show also that c is differentiable on a set of measure 1.

Problem 8. *Use the function c defined in the previous question to give an example of a continuous function f and a measureable set A such that $f(A)$ is not measureable.*