1. Problem Set 4

Problem 1. *Suppose* $u(x)$ *is a nonnegative integrable function and* $a \in \mathbb{R}$. *Let* $u_a(x) = u(x-a)$. *Show that* $\int_{\mathbb{R}} u_a = \int_{\mathbb{R}} u$.

Problem 2. *Suppose* f *is integrable. Show that there exists a sequence* $\{f_k\}_{k\in\mathbb{N}}$ *of integrable simple functions such that* $f_k \to f$ *pointwise.*

Problem 3. *Suppose* f *is integrable, and let*

$$E_a = \{x \in \mathbb{R} | f(x) > a\}.$$

Show that

$$m(E_a) \leq \frac{1}{a} \int |f|$$

for all $a > 0$.

Problem 4. *Find a measureable function* f *for which*

$$m(E_a) \leq \frac{1}{a}$$

for all $a > 0$, *and* f *is not integrable.*

Problem 5. *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is measureable, and Riemann integrable on* $[a, b]$. *Show that the Lebesgue integral* $\displaystyle\int_{[a,b]} f$ *exists and is equal to the Riemann integral* $\displaystyle\int_a^b f(x)dx$.

Problem 6. *Suppose* $f([a, b] \to \mathbb{R}$ *is measureable, and Riemann integrable on* $[x, b]$ *for each* $a < x < b$. *Show that the Lebesgue integral* $\displaystyle\int_{[a,b]} f$ *exists and*

$$\int_{(a,b]} f = \lim_{x\to a} \int_x^b f(x)dx$$

Use this fact to provide an explicit example of a Lebesgue integrable function on $[0, 1]$ *which is not bounded.*

Problem 7. *Suppose* f *is continuous and integrable. Does it follow that*

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0?$$

Problem 8. *Suppose* f *is integrable. Does it follow that there exists an open interval* (a, b) *on which* f *is bounded?*