1. Problem Set 5

Problem 1. *Suppose f is integrable. Show that*

$$\lim_{n\to\infty} \int_{[} -1/n, 1/n]f = 0.$$

Show that this is not true in general if f is only measurable.

Problem 2. *Recall from Problem Set 3 that if $\{f_n\}$ are a sequence of measurable functions that converge pointwise to f on a set A of finite measure, then for any $\varepsilon > 0$ there exists a measurable $B \subset A$ so that $m(A \setminus B) < \varepsilon$ and $f_n \to f$ uniformly on B. This is often called* Egorov's theorem. *Use Egorov's theorem to provide a direct proof that if $\{f_n\}$ are a sequence of measurable functions that converge pointwise to f on a set A of finite measure, and $|f_n(x)| \leq M$ for each $n \in \mathbb{N}, x \in A$, then*

$$\lim_{n\to\infty} \int f_n = \int f.$$

Problem 3. *Find a sequence of nonnegative measurable functions $\{f_n\}$ such that*

$$\int \liminf f_n < \liminf \int f_n.$$

Problem 4. *Suppose f is integrable, g is differentiable, and both g and its derivative are bounded. Show that $f * g$ is differentiable, and*

$$(f * g)'(x) = f * g'(x).$$

Problem 5. *Suppose f is integrable. Show that $f * \chi_{(0,\infty)}$ is continuous.*

Problem 6. *Suppose f is integrable. Show that $f * \chi_{(0,\infty)}$ is uniformly continuous.*

Problem 7. *Suppose f is integrable, and let*

$$E_a = \{x \in \mathbb{R} | f(x) > a\}.$$

Show that

$$\lim_{a\to\infty} am(E_a) = 0.$$

Problem 8. *Suppose f is integrable and $\int_I f = 0$ for every interval I. Show that there exists a set E of measure zero such that $f = 0$ on $\mathbb{R} \setminus E$.*