1. Problem Set 6

Problem 1. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Show that the graph of f has measure zero in $\mathbb{R}^2$: that is, show that the set*

$$\textit{graph}(f) = \{(x, f(x)) | x \in \mathbb{R}\}$$

has measure zero.

Problem 2. *Does the result of Problem 1 hold for arbitrary functions $f : \mathbb{R} \to \mathbb{R}$?*

Problem 3. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is integrable. Let $a > 0$ and define $f_a(x) = f(ax)$. Show that*

$$a^n \int_{\mathbb{R}^n} f_a = \int_{\mathbb{R}^n} f.$$

Problem 4. *Suppose $f : \mathbb{R}^n \to \mathbb{C}$ is an integrable function. Define the Fourier transform of f by*

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx.$$

Show that $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ is bounded and continuous. (Here i is the imaginary number $\sqrt{-1}$, but the complex character of the function shouldn't bother you too much as long as you know that $e^{ix} = \cos(x) + i \sin(x)$.)

Problem 5. *Suppose f and g are integrable functions on $\mathbb{R}^n$. Show that $f(x - y)g(y)$ is integrable on $\mathbb{R}^{2n}$. Use this fact to show that the convolution*

$$f * g(x) = \int_{\mathbb{R}^n} f(x - y) g(y) dy$$

is defined for almost every $x \in \mathbb{R}^n$.

Problem 6. *Suppose f and g are integrable functions on $\mathbb{R}^n$. Show that the convolution $f * g$ is integrable.*