1. Problem Set 7

Problem 1. *Suppose $f : \mathbb{R}^{n_1} \to \mathbb{R}$ is measurable, and define $F : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ by*

$$F(x, y) = f(x).$$

Show that F is measurable.

Problem 2. *Suppose $f : \mathbb{R}^{n_1} \to \mathbb{R}$ and $g : \mathbb{R}^{n_2} \to \mathbb{R}$ are integrable functions, and define $F : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ by*

$$F(x, y) = f(x)g(y).$$

Show that F is integrable.

Problem 3. *Suppose f and g are integrable functions on $\mathbb{R}^n$. Show that $f(x-y)g(y)$ is integrable on $\mathbb{R}^{2n}$. Use this fact to show that the convolution*

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

is defined for almost every $x \in \mathbb{R}^n$.

Problem 4. *Suppose f and g are integrable functions on $\mathbb{R}^n$. Show that the convolution $f * g$ is integrable.*

Problem 5. *Suppose f and g are integrable on $\mathbb{R}^n$. Show that*

$$\lim_{x \to \infty} f * g(x) = 0.$$

Recall from the previous problem set that if f is integrable, then Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi}\, dx$$

is defined, bounded, and continuous.

Problem 6. *Show that $\widehat{f * g} = \hat{f}\hat{g}$.*

Problem 7. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is an integrable function. Show that there exists a sequence of functions $\{f_k\}$ of the form*

$$f_k = \sum_{n=1}^{N} a_k \chi_{U_k},$$

where U_k are open sets of finite measure, such that $f_k \to f$ pointwise, except possibly on a set of measure zero.

Problem 8. *Show that*

$$\lim_{|\xi| \to \infty} \hat{f}(\xi) = 0$$

(This is the Riemann-Lebesgue lemma.)