1. Problem Set 8

Problem 1. *Find a sequence of functions* $\{f_k\} \subset L^1(\mathbb{R})$ *such that* $f_k \to f$ *in* L^1 *but* $f_k(x)$ *does not converge to* $f(x)$ *at any* x.

Problem 2. *Suppose* $1 \leq p, q < \infty$. *Show that* $L^p(\mathbb{R})$ *is contained in* $L^q(\mathbb{R})$ *if and only if* $p = q$.

Problem 3. *Define* $C_c^k(\mathbb{R}^n)$ *to be the set of compactly supported continuous functions on* $\mathbb{R}^n$ *whose* j^{th} *partial derivatives are continuous, for* $1 \leq j \leq k$. *Show that* $C_c^k(\mathbb{R}^n)$ *is dense in* $L^1(\mathbb{R}^n)$.

Problem 4. *Suppose* $f, g \in L^1(\mathbb{R})$. *Define* $F(x) = \int_{-\infty}^x f$ *and* $G(x) = \int_{-\infty}^x g$. *(Why are these defined?) Show that*

$$\int_a^b Fg = F(b)G(b) - F(a)G(a) - \int_a^b fG.$$

Problem 5. *Suppose* $f : \mathbb{R}^n \to \mathbb{R}$. *Define the essential supremum of* f *by*

$$ES(f) = \inf\{a | m(|f|^{-1}(a, \infty)) = 0\}.$$

Show that if $|f(x)| < M$ *almost everywhere, then* $ES(f)$ *exists and is bounded above by* M.

Problem 6. *Suppose* $f : \mathbb{R}^n \to \mathbb{R}$ *is integrable and* $|f(x)| < M$ *almost everywhere. Show that* $f \in L^p(\mathbb{R}^n)$ *for all* $p > 1$,

$$\lim_{p \to \infty} \left(\int_{\mathbb{R}^n} |f|^p \right)^{1/p} = ES(f).$$

Problem 7. *The previous problem justifies the usual definition of* $L^\infty(\mathbb{R}^n)$:

$$L^\infty(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R} | ES(f) < \infty\} / f \sim g \text{ if } f = g \text{ a.e.}$$

with the norm

$$\|f\|_{L^\infty(\mathbb{R}^n)} = ES(f)$$

and the distance

$$d(f, g) = \|f - g\|_{L^\infty(\mathbb{R}^n)}.$$

Show that $L^\infty(\mathbb{R}^n)$ *is a metric space.*