1. Problem Set 9 – Fourier Series

This problem set is dedicated to proving the following claim: the set $\{e^{inx}\}_{n\in\mathbb{Z}}$ forms an orthonormal basis of $L^2[-\pi,\pi]$. In particular, any $f \in L^2[-\pi,\pi]$ can be written (uniquely) as a sum

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$$

with

$$\|f\|_{L^2[-\pi,\pi]} = \|\{a_n\}\|_{\ell^2}.$$

This is an incredibly useful fact to know for solving differential equations, since derivatives (and integrals) of e^{inx} are so easy to calculate.

In this problem set we will define $L^2[-\pi,\pi]$ to be the set of *complex valued* functions for which $|f(x)|^2$ is integrable. You may need to use the fact that $C_c^1[-\pi,\pi]$ is dense in $L^p[-\pi,\pi]$ for $1 \le p \le \infty$.

Problem 1. *Let $m,n \in \mathbb{Z}$. Define the Kronecker delta, δ_{mn}, to be 1 when $m = n$ and zero otherwise. Prove that*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{imx}e^{-inx}dx = \delta_{mn}.$$

You can interpret this from a linear algebra point of view: if we define the inner product of two functions f,g on the interval $[-\pi,\pi]$ by the integral

$$(f,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g}(x)dx,$$

then what we have just shown is that the set of all $e^{inx}, n \in \mathbb{Z}$, forms an orthonormal set.

Problem 2. *Suppose that $f(x) = \sum_{n=-N}^{N} a_n e^{inx}$, where $a_n \in \mathbb{C}$. Show that*

$$\|f\|^2_{L^2([-\pi,\pi])} = 2\pi \sum_{n=-N}^{N} |a_n|^2,$$

and moreover, that

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

Problem 3. *Now suppose $f \in L^1([-\pi,\pi])$. Define*

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx,$$

and show that

$$\lim_{n\to\pm\infty} a_n = 0.$$

(Hint: first do this for $f \in C_c^1$, and then use the density fact stated in the introduction. The result of this problem is sometimes called the Riemann-Lebesgue lemma.)

Problem 4. *Suppose $f \in C_c^1([-\pi,\pi])$, and define a_n as in the previous question. Define*

$$S_N(x) = \sum_{n=-N}^{N} a_n e^{inx}.$$

Show that for each $x \in (-\pi,\pi)$,

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x-y)f(y)dy$$

where

$$K_N(\theta) = \frac{\sin((N+\frac{1}{2})\theta)}{\sin(\frac{1}{2}\theta)}$$

Hint: Use Fubini, and sum a geometric series.

Problem 5. *Show that K_N is continuous, and*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta)d\theta = 1.$$

(Hint: use the series form of K_N.) Now show that for $f \in C_c^1([-\pi,\pi])$,

$$\begin{aligned} S_N(x) - f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta)[\tilde{f}(x+\theta) - \tilde{f}(x)]d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin((N+1/2)(\theta))\left[\frac{\tilde{f}(x+\theta)-\tilde{f}(x)}{\sin(\frac{1}{2}\theta)}\right]d\theta \end{aligned}$$

for each $x \in (-\pi,\pi)$. Here $\tilde{f}$ is the periodic extension of f to the whole real line.

Problem 6. *Show that for $f \in C_c^1([-\pi,\pi])$, we have $S_N \to f$ uniformly on $[-\pi,\pi]$. Hint: think about your reasoning for question 3.*

Problem 7. *Now show that for $f \in L^2([-\pi,\pi])$,*

$$\lim_{N\to\infty} \|S_N - f\|_{L^2([\pi,-\pi])} = 0.$$

Problem 8. *Reinterpreting the previous problem, we have*

$$f(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx}$$

in the L^2 sense, where

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

Show that $\{a_n\} \in \ell^2$, and

$$\|\{a_n\}\|_{\ell^2} = \|f\|_{L^2[-\pi,\pi]}.$$