1. Open Problems

1.1. The Anisotropic Calderon Problem.

In the classical Calderón problem, we have an unknown positive conductivity function $\gamma : \Omega \to \mathbb{R}$. This defines the Dirichlet-Neumann map $\Lambda_\gamma : f \mapsto \gamma \partial_\nu u$, where u solves

$$\begin{aligned} \nabla \cdot \gamma \nabla u &= 0 \\ u|_{\partial\Omega} &= f \end{aligned}$$

If we regard $\nabla \cdot \gamma \nabla u$ as the divergence of a vector field $\gamma \cdot \nabla u$ obtained by taking the gradient of u and transforming it by multiplication by γ, then we can take the point of view that there's no reason to stop at scalar multiplication. Instead we can ask about the Dirichlet-Neumann map Λ_A defined by the operator

$$\begin{aligned} \nabla \cdot A \nabla u &= 0 \\ u|_{\partial\Omega} &= f \end{aligned} \tag{1.1}$$

where A is a matrix valued function $A : \Omega \to \mathrm{GL}(n, \mathbb{R})$. In the case where A is a positive definite symmetric matrix, the operator $\nabla \cdot A \nabla u$ is a well behaved elliptic operator, and this is a natural generalization of the ordinary Calderón problem.

If this isn't enough of a reason to study the problem, this also has a natural physical interpretation. It corresponds to the case where electric conductivity is anisotropic – where conductivity is better in some directions than others. This is in fact the case in many applications we want to study – muscles conduct electricity better down the fibers than across them, sedimentary rocks conduct better along the layers than across them, etc.

There's another point of view as well. If we have a Riemannian manifold (M, g), the Laplacian has a natural analogue called the Laplace-Beltrami operator $\triangle_g$, given in coordinates by

$$\triangle_g u = |g|^{-\frac{1}{2}} \nabla \cdot (|g|^{\frac{1}{2}} g^{-1} \nabla u).$$

If M has a boundary, the Laplace-Beltrami equation

$$\begin{aligned} \triangle_g u &= 0 \\ u|_{\partial M} &= f \end{aligned}$$

gives rise to a natural Dirichlet-Neumann map Λ_g on ∂M, and we can ask if Λ_g determines the metric g. If you stare at this problem long enough, you should be able to convince yourself that the anisotropic Calderón problem is a special case of this. (They're not equivalent, because the anisotropic Calderón problem needs the manifold to be given by a single coordinate chart, whereas in general this doesn't need to be the case.)

So we want to figure out if Λ_g determines g, but we should immediately be suspicious: in a positive symmetric matrix g, we need to recover $n(n+1)/2$ functions from the same amount of boundary data that we had in the Calderón problem. Sure enough, identifiability does not hold in this problem.

To see this, consider a diffeomorphism $\psi : M \to M$ with the property that $f(x) = x$ for all x in a neighbourhood of the boundary. We define the pullback

$$\psi^* u(x) = u(\psi(x))$$

and check that if u satisfies (1.1), then $\psi^* u(x)$ satisfies an equation

$$\triangle_{\tilde{g}} \psi^* u = 0$$
$$\psi^* u|_{\partial\Omega} = f$$

where $\tilde{g}$ is a (different) Riemannian metric. Moreover $\psi^* u = u$ in a neighbourhood of the boundary, which means that g and $\tilde{g}$ have the same Dirichlet-Neumann map.

Therefore the best we can hope for is that we can recover g from Λ_g up to a boundary-preserving diffeomorphism. The basic version of the anisotropic Calderón problem is this: given a Riemannian manifold (M, g) with boundary, does the map Λ_g determine g up to boundary-preserving diffeomorphism? This is an open problem.

If you're unsatisfied with the equivalence relation here, it might help to know that if two metrics differ by a conformal factor – i.e.

$$g_1 = c(x) g_2,$$

where c is uniformly positive, then showing that g_1 is equal to g_2 up to a boundary preserving diffeomorphism shows that $g_1 = g_2$. So an easier version of the problem is the following: given a fixed Riemannian metric g, and two Riemannian metrics $g_1 = c_1 g$ and $g_2 = c_2 g$, does

$$\Lambda_{g_1} = \Lambda_{g_2}$$

imply that $g_1 = g_2$? If c_1 and c_2 are known to be C^2, then by the same type of change of variables used for the classical Calderón problem, this reduces to the following problem. Suppose we are given a fixed Riemannian metric g, and consider the equation

$$(\triangle_g + q) u = 0.$$

This defines a Dirichlet-Neumann map Λ_q, and we can ask whether $\Lambda_{q_1} = \Lambda_{q_2}$ implies that $q_1 = q_2$. This is also an open problem – let's call it the conformal class problem – and we'll concentrate on this version here, since it seems to be a fair bit easier.

1.2. **An inadequate idea.** A first approach would be to try to imitate the proof of the classical case as far as possible. So to start, we'd use the integration by parts find that

$$\int_M (q_1 - q_2) u_1 u_2 = \int_{\partial M} (\Lambda_{q_1} - \Lambda_{q_2}) u_1 u_2$$

whenever $(\triangle_g + q_j) u_j = 0$. Then we'd conclude that if $\Lambda_{q_1} = \Lambda_{q_2}$ then

$$\int_M (q_1 - q_2) u_1 u_2 = 0$$

whenever $(\triangle_g + q_j) u_j = 0$. So far so good. Now we'd want to construct CGO solutions u_1 and u_2 like we did in the classical case, and now things start to fall apart. The key is that we want to build solutions of the form

$$u_1 = e^{\tau\varphi}(a_1 + r_1), \quad u_2 = e^{-\tau\varphi}(a_2 + r_2)$$

where a_1 and a_2 are not too large, and r_1, r_2 are actually small as $\tau \to \infty$. One can show that this is equivalent to asking for a Carleman estimate for both the weights φ and $-\varphi$.

Sadly, Dos Santos Ferreira, Kenig, Salo, and Uhlmann showed in a 2008 paper ("Limiting Carleman weights and anisotropic inverse problems") that this is only possible if g can be written in the form

$$g = c(x) \begin{pmatrix} 1 & 0 \\ 0 & g' \end{pmatrix}$$

where g' is a Riemannian metric on an $n-1$ dimensional manifold. That is, the anisotropy has to leave one direction alone – the manifold M is only transversally anisotropic. Clearly this isn't true of an arbitrary metric, and now we're stuck.

1.3. **Connection to X-ray problems.** One interesting fact highlighted in (but not original to) the Dos Santos Ferreira-Kenig-Salo-Uhlmann paper is the connection between this problem and the geodesic X-ray problem.

You can see this connection already in the Calderón problem. The reason we can build the CGO solution

$$u(x) = e^{\tau x_1}(e^{i\tau x_2} + r(x))$$

is that if you plus this into

$$(\triangle + q)u = 0$$

you find that r must satisfy

$$(e^{-\tau x_1} \triangle e^{\tau x_1} + q)r = -q e^{i\tau x_2} = O(1),$$

and we have a Carleman estimate telling us how to invert the operator on the left to get $r \sim O(\tau^{-1})$.

But from this point of view there's nothing super special about the function $e^{i\tau x_2}$: any function a with the property that

$$\triangle e^{\tau x_1} a = O(1)$$

will work. In particular, you could take

$$a = e^{i\tau x_2}\alpha(x_3)$$

for any C^2 function α. Then you might consider choosing α to be really concentrated near the plane $x_3 = c$, and obtain a CGO solution with almost all of its mass supported near the $x_3 = c$ plane.

Using such CGO solutions you could imagine reducing the ordinary Calderón problem to a Radon-type problem, and indeed in the DKSU paper, the authors solve the conformal class problem only with the additional hypothesis that the transverse manifold with metric g' must be such that the geodesic ray transform is invertible.

Much of the effort in improving this result have gone into increasing the class of manifolds on which the geodesic ray transform is invertible, which is of course a highly interesting problem in its own right.

Eliminating the transversally anisotropic condition seems much less tractable and the original problem remains unsolved.

2. Non-Ohmic materials

Another important unsolved problem regards non-ohmic resistors. In writing down the Calderón problem, we implicitly made the assumption that the current is a multiple of the electric field ∇u, with some mulitplier γ: that is to say that the current field J is given by

$$J = \gamma \nabla u.$$

This is what we would expect from Ohm's law: the current is proportional to the differential of the voltage, with the proportion being the conductivity, or the inverse of the resistivity.

On the other hand not all materials and regimes obey Ohm's law: a more general rule would be that the current depends on some function of the electric field, so

$$J = \gamma(x, |\nabla u|)\nabla u.$$

We would still expect the current to be divergenceless – that's an actual physical law – so the equation to consider would be

$$\begin{aligned} \nabla \cdot \gamma(x, |\nabla u|)\nabla u &= 0 \\ u|_{\partial\Omega} &= f. \end{aligned}$$

We would expect this to define a voltage-to-current map $\Lambda_\gamma : f \mapsto \gamma(x, |\nabla u|)\partial_\nu u|_{\partial\Omega}$ and we can ask if this determines γ.

This is a pretty hard problem. A simple dimension counting argument should make you skeptical that this is even reasonable, and the problem is worse than that: it's not necessarily easy to understand the set of γ for which the voltage-to-current map is even defined.

We can make things easier by considering a simpler version: we'll suppose that the conductivity function takes the form $\gamma(x)|\nabla u|^{p-2}$. Then we have the equation

$$\begin{aligned} \nabla \cdot \gamma(x)|\nabla u|^{p-2}\nabla u &= 0 \\ u|_{\partial\Omega} &= f. \end{aligned} \tag{2.1}$$

This defines a voltage-to-current map $\Lambda_\gamma : f \mapsto \gamma(x)|\nabla u|^{p-2}\partial_\nu u|_{\partial\Omega}$. The question is whether Λ_γ determines γ. A quick look at some graphs of properties for non-ohmic materials confirms that this model, with $p > 2$, is a reasonable model for a range of materials, and as a bonus, the underlying operator for $\gamma \equiv 1$ is the p-Laplacian, a well-studied nonlinear operator.

Needless to say this problem is wide open.

2.1. An approach.

Let's try to solve it anyway. In analogy with the ordinary Calderón problem, the first thing we could think of doing is an integration by parts. If u solves (2.1), we have

$$0 = \int_\Omega \nabla \cdot \gamma|\nabla u|^{p-2}\nabla u \, \bar{v} dx,$$

and integrating by parts, we get

$$\int_{\partial\Omega} \Lambda_\gamma(u)\bar{v}\, dS = \int_\Omega \gamma |\nabla u|^{p-2}\nabla u \cdot \nabla v dx.$$

We can't do the usual tricks of looking at the difference of two solutions, because the whole problem is hopelessly nonlinear. But at least this gives hope that if we know the voltage and current at the boundary, that we can recover γ.

To do this it would appear that we need an analog of the CGO solutions. Amazingly this is at least a little plausible too: Tom Wolff noted in the 80s (in a paper that was published in 2007!) that exponential functions $e^{\alpha+i\beta)\cdot x}$ solve the $p - Laplacian$ as long as

$$(p-1)|\alpha|^2 = |\beta|^2 \text{ and } \alpha \cdot \beta = 0.$$

With the right choice of v you can just about picture making the right side of the integral inequality into a Fourier transform of γ.

The real problem is that there's no clear way to modify a Wolff exponential solution in a "small" way to get a solution to (2.1).

Adding a small correction is probably the wrong idea in the context of a nonlinear problem; multiplying by something close to one is more reasonable, but the real problem is that we have very few handles on the p-Laplace operator in general.

One major demonstration of this is that in $n \geq 3$ dimensions, the question of whether unique continuation holds for the p-Laplace operator is unknown. That is, if $\nabla\cdot|\nabla u|^{p-2}\nabla u = 0$ in $\mathbb{R}^n$ and $u = 0$ in an open set Ω, it is unknown whether this implies that $u \equiv 0$.

A moment's reflection should persuade you that this means the inverse problem is hard: if we can't even determine the solution at one point from its behavior in an open set, the problem of recovering a coefficient is going to be hard.

Even solving the unique continuation problem for the p-Laplacian would be a big breakthrough.

2.2. **Alternate problems.** One alternative is to try to solve less aggressively nonlinear problems. In a large class of quasi and semi-linear problems, Isakov and Sun showed (in separate papers) that the Dirichlet-Neumann maps for the nonlinear problem can be used to recover the Dirichlet-Neumann map for a corresponding linear problem, which is solvable. But this approach does not seem to work for the p-Laplace equation.

Another approach is to solve alternate problems for the p-Laplacian. Salo and Zhong showed that boundary values of γ could be recovered from the voltage-current map of the p-Laplace problem, and Brander extended this to show that the first derivatives of γ could also be recovered. In the two-dimensional case, slightly more is known. As far as I know, no one has tried to tackle the discrete version of this problem. But none of these approaches appear to have much bearing on the full problem.

3. Partial Data Problems

Another set of open problems concerns partial data. In the classical Calderón problem, we want to find γ assuming we know Λ_γ on the whole boundary. In practice, though, measuring Λ_γ on the whole boundary is impossible or impractical. It would be nice to know if we can measure Λ_γ on part of the boundary only and still recover γ.

To state this problem clearly, it helps to pin down what we mean by "measure Λ_γ on part of the boundary only." The clearest thing we could mean is that there is some subset $\Gamma \subset \partial\Omega$, and we want to know if

$$\{\Lambda_\gamma(f)|_\Gamma | f \in H^{\frac{1}{2}}(\partial\Omega)\}$$

determines γ. This is called the partial output problem.

Upon further reflection, though, this problem isn't that great: the experimenters still have to impose a boundary potential f which may be supported on the whole boundary, and to do this, they need access to the whole boundary. A more restrictive version of the partial data problem is to fix some subset $\Gamma \subset \partial\Omega$, and ask if knowledge of

$$\{\Lambda_\gamma(f)|_\Gamma | f \in H^{\frac{1}{2}}(\partial\Omega), f|_{\partial\Omega\setminus\Gamma} = 0\}$$

suffices to recover γ. In other words, you know $\Lambda_\gamma(f)$ only for f which is grounded on the inaccessible side. (If you don't like this problem, there's an alternate version with current-to-voltage maps where the object is insulated on the inaccessible side.) Let's call this the local Calderón problem.

3.1. **Partial Output Problem.** At this point the partial output problem is fairly well (although not completely) understood. Kenig, Sjöstrand, and Uhlmann proved in 2008 that if Ω is a bounded smooth convex domain, then the partial output problem is solvable for any open subset $\Gamma \subset \partial\Omega$.

Here's a brief sketch of their proof:

By the usual argument, they get the integral equality

$$\int_\Omega (q_1 - q_2)u_1 u_2 \, dx = \int_{\partial\Omega} (\Lambda_{q_1} - \Lambda_{q_2})(u_1)u_2 \, dS$$

If we know that $\Lambda_{q_1} = \Lambda_{q_2}$ on Γ, then we get

$$\int_\Omega (q_1 - q_2)u_1 u_2 \, dx = \int_{\Gamma^c} (\Lambda_{q_1} - \Lambda_{q_2})(u_1)u_2 \, dS$$

As usual, we'd like to use CGO solutions with cancelling real exponential parts for u_1 and u_2. To make the right side disappear, though, you need to make u_2 vanish on Γ^c.

How do you ensure this? Recall that we built CGO solutions of the form

$$u = e^{\tau\varphi}(a + r)$$

The error term r is built with a Carleman estimate, and the Carleman estimate is proved using an integration by parts. When we did the Carleman estimate earlier, we used functions that were compactly supported, to avoid pesky boundary terms. But it turns out that if you relax this condition, you end up with boundary terms like

$$(\partial_\nu\varphi\partial_\nu u, \partial_\nu u),$$

just as we did in the control theory case. When $\partial_\nu\varphi$ has the right (negative) sign, this term ends up being bounded above, and the resulting Carleman estimate looks something like this:

$$\tau\|\sqrt{|\partial_\nu\varphi|}\partial_\nu u\|^2_{\Gamma^c}\tau\|u\|^2_\Omega \lesssim \|\triangle_\varphi u\|^2_\Omega,$$

where Γ^c is the set on which $\partial_\nu\varphi$ is negative. If you plug this Carleman estimate into the duality argument that gives you r, you find that the resulting solutions can be defined on Γ^c.

Bukhgeim and Uhlmann used this strategy to show that CGO solutions can be made to vanish on part of the boundary without messing with the principal terms, and used this to complete the argument. Kenig, Sjöstrand, and Uhlmann improved this strategy essentially by picking Carleman weights better, and then in the Dos Santos Ferriera-Kenig-Salo-Uhlmann paper referenced earlier, they showed essentially that this is as good as it gets.

3.2. **The Local Problem.** We could try to apply this logic to the local problem: if u_2 can be made to vanish on the set where $\partial_\nu\varphi$ is negative, then what about u_1? Because u_1 and u_2 are built with cancelling Carleman weights, u_1 and u_2 can only vanish on opposite sides of $\partial\Omega$. If you look at the integral equality

$$\int_\Omega (q_1 - q_2)u_1u_2\,dx = \int_{\partial\Omega}(\Lambda_{q_1} - \Lambda_{q_2})(u_1)u_2\,dS,$$

you can see this is no good for the local problem: if you want to eliminate the right hand side you need u_1 and u_2 to vanish on the same set.

Kenig and Salo proved a result that says that you can solve the local problem, roughly speaking, if the inaccessible portion is flat in one direction: this allows them to put the entire inaccessible set into the part of the boundary where $\partial_\nu\varphi = 0$, which is the limiting case.

But the Dos Santos Ferriera-Kenig-Salo-Uhlmann which lists all the available Carleman weights in Euclidean space, says that this strategy cannot be taken further.

Meanwhile Isakov independently used a reflection argument to solve the case where the inaccessible part lies in a hyperplane or hypersphere, but this also does not generalize.

Ideally one would hope to prove an analog of the Kenig-Sjöstrand-Uhlmann result for the local problem, but this remains an open problem.