Preliminaries and Background

In these notes the notation ∂_x^α should be interpreted as follows: α is a multiindex $\alpha = (\alpha_1, \ldots \alpha_n)$, with each α_i being a nonnegative integer. Then

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}.$$

In this context $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

1. Spaces

Most of this course concerns functions on a domain $\Omega \subset \mathbb{R}^n$. Unless otherwise stated, we will generally assume that Ω is a bounded open set in $\mathbb{R}^n$ with smooth boundary.

Most of the theorems in this course concern one of the following function spaces:

1.1. C^k **Spaces.** (k-times differentiable functions)

We define $C^k(\Omega)$ to be the space of functions on Ω which are k times differentiable and whose k^{th} derivatives are continuous. The space $C^k(\Omega)$ has the natural norm

$$\|f\|_{C^k(\Omega)} = \sum_{|\alpha| \leq k} \sup_{\Omega} |\partial_x^\alpha f(x)|.$$

The notation $C^\infty(\Omega)$ refers to the space of smooth (infinitely differentiable) functions on Ω. It has a natural metric too (exercise: what is it?) but people don't normally think of it this way.

The notation $C_0^k(\Omega)$ refers to the subspace of $C^k(\Omega)$ consisting of functions f for which there is a compact set $K \subset \Omega$ such that $f \equiv 0$ outside of K; you can think of $C_0^k(\Omega)$ functions being those that vanish in a neighbourhood of the boundary.

In this course we will mostly concern ourselves with the case in which k is a nonnegative integer. (C^α with non-integer alpha usually refers to a Hölder space. Hölder spaces play an important role in the theory of inverse problems, but most of this theory lies beyond the scope of this course.)

Note that the derivative operator gets along well with C^k spaces, as long as k is big enough: a derivative operator D^a of order a maps $C^k(\Omega)$ to $C^{k-a}(\Omega)$, provided that $k \geq a$.

1.2. L^p **Spaces.** ($|f|^p$ is integrable)

The space $L^p(\Omega)$ is usually defined for $1 \leq p < \infty$ as the space of all measurable functions f on Ω such that

$$\int_\Omega |f(x)|^p dx < \infty,$$

modulo the equivalence relation $f \simeq g$ if $f = g$ almost everywhere. It has the natural norm

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx \right)^{1/p}.$$

$L^\infty(\Omega)$ is more complicated (because of those weasel words "almost everywhere") but we can think of it as the set of all bounded measurable functions on Ω with the norm

$$\|f\|_{L^\infty(\Omega)} = \inf_{g=f\text{a.e.}} \sup\{|g(x)||x \in \Omega\}.$$

For almost all of this course we will concern ourselves only with the three best L^p spaces: L^1, L^2, and L^∞. It's important to note that

- $C^k(\Omega)$ is dense in $L^p(\Omega)$ for any $1 \le p < \infty$ and nonnegative integer k.
- Since $L^p(\Omega)$ is also complete, we could think of $L^p(\Omega)$ as the *completion* of $C^\infty(\Omega)$ in the L^p norm. That is, we can think of $L^p(\Omega)$ as the space of all functions which are the limit (in the L^p norm sense) of a sequence of $C^\infty(\Omega)$ functions.
- $L^2(\Omega)$ is a Hilbert space with inner product $\langle f, g\rangle = \int_\Omega fg$.

Note that the derivative operator does not get along well with L^p spaces: in general an L^p function is not differentiable in the classical sense. For differentiation we need to pass to the notion of...

1.3. **Sobolev Spaces.** (k weak derivatives are in L^p)

We want to define the Sobolev space $W^{k,p}$ to be the space of functions whose k^{th} derivatives are L^p. Unfortunately if we do this, then the resulting space is no longer complete.

To fix this we define the weak derivative: $w = \partial_x^\alpha u$ in the weak sense if

$$(-1)^{|\alpha|} \int_\Omega w \partial_x^\alpha \varphi = \int_\Omega u\varphi$$

for every $\varphi \in C_0^\infty(\Omega)$. We can define the Sobolev space $W^{k,p}(\Omega)$ as the space of all functions in L^p for which all weak derivatives up to order k exist and are L^p. It has the natural norm

$$\|f\|^p_{W^{k,p}(\Omega)} = \sum_{|\alpha|\le k} \|\partial_x^\alpha f\|^p_{L^p(\Omega)}$$

As with the L^p spaces, it turns out that $C^k(\Omega)$ is dense in $W^{k,p}(\Omega)$ and so we can think of $W^{k,p}(\Omega)$ as the completion of $C^k(\Omega)$ under the $W^{k,p}(\Omega)$ norm.

The $W^{k,2}$ spaces are the best ones, and they are often written H^k, where H stands for Hilbert space. They have a natural inner product which we will mostly ignore.

For our purposes one of the most striking facts about Sobolev spaces is that they have a trace property: for example, if $u \in H^k(\Omega)$ for $k \ge 1$, then $u|_{\partial\Omega} \in H^{k-1}(\partial\Omega)$. This is not true for L^p functions – they're only defined almost anywhere – but functions in sufficiently good Sobolev spaces have enough regularity that the restriction to smaller-dimension sets can make sense. (Here $f \in H^{k-1}(\partial\Omega)$ should be interpreted as follows: if we take a section of $\partial\Omega$ and represent it as a graph of a function g, then $f \circ g$ should be in $H^{k-1}(\mathbb{R}^{n-1})$.

Actually $u|_{\partial\Omega} \in H^{k-\frac{1}{2}}(\Omega)$, but to make sense of this statement we will have to read more below.

2. Tools

Most of this course uses three main tools, aside from the standard stuff one learns in undergraduate multivariable calculus/ ODE courses.

Anything more complicated than this we'll do in detail.

2.1. Integration by Parts.

Ok, so people do learn about this one in undergraduate courses, but usually no one talks about how critically important it is.

Typically we'll use a multivariable version of integration by parts, since inverse problems are not usually any fun in one dimension. The most important version is the divergence (or Gauss's) theorem: for a vector field $V : \mathbb{R}^n \to \mathbb{R}^n$,

$$\int_\Omega \nabla \cdot V dx = \int_{\partial\Omega} \nu \cdot V dS,$$

where ν is the (outward) unit normal vector on $\partial\Omega$. This has two important consequences for the Laplacian $\triangle u = \nabla \cdot \nabla u$:

$$\int_\Omega u \triangle v dx = -\int_\Omega \nabla u \cdot \nabla v dx + \int_{\partial\Omega} u \partial_\nu v dS$$

and

$$\int_\Omega \triangle u v dx - \int_\Omega v \triangle u dx = \int_{\partial\Omega} v \partial_\nu u dS - \int_{\partial\Omega} u \partial_\nu v dS.$$

These two statements are sometimes called Green's first and second identities.

The divergence theorem also has an important consequence for the partial derivative $\partial_{x_i} = \nabla \cdot e_i$:

$$\int_\Omega \partial_{x_i} u v dx = -\int_\Omega u \partial_{x_i} v dx + \int_{\partial\Omega} \nu_i u v dS,$$

where ν_i is the i^{th} component of ν.

2.2. Fourier Transform.

For $f \in L^1(\mathbb{R}^n)$, we define

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx.$$

A priori the Fourier transform maps $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$, but it turns out that it can be extended to a map from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$, with the Plancherel identity

$$\|\hat{f}\|_{L^2} = (2\pi)^? \|f\|_{L^2}.$$

We will mostly consider the Fourier transform as a map from L^2 to L^2. The Fourier transform has an inverse transform

$$\check{f}(\xi) = (2\pi)^? \int_{\mathbb{R}^n} f(x) e^{ix\cdot\xi} dx.$$

The Fourier transform has a number of cute properties but for our purposes the most important is the following:

$$\widehat{\partial_x^\alpha f}(\xi) = (i\xi)^\alpha \hat{f}(\xi).$$

For linear equations, this can turn the problem of solving PDE into an algebraic problem! For example, if we have

$$(1-\triangle)u = f$$

then

$$(1+|\xi|^2)\hat{u} = \hat{f}$$

so

$$\hat{u} = \frac{1}{1+|\xi|^2}\hat{f}.$$

The Fourier transform also has an important role in understanding Sobolev spaces. It turns out that for nonnegative integer k,

$$\|u\|_{H^k(\mathbb{R}^n)} \simeq \|(1+|\xi|)^k \hat{u}\|_{L^2(\mathbb{R}^n)}.$$

This provides a convenient way of understanding $H^k(\mathbb{R}^n)$ for any k: just use the above identity!

2.3. **Neumann Series.** Also called Born series, a Neumann series is an operator version of the following Taylor series fact: for $|x| < 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots$$

Suppose we have an operator $K : X \to X$, such that there exists $c < 1$ so that

$$\|Ku\|_X \leq c\|u\|_X.$$

It turns out that the operator $I - K$ is invertible and

$$(I-K)^{-1} = I + K + K^2 + K^3 + \ldots$$

This is easy to prove: the right side convergence because of the convergence of geometric series, and if one applies $(I-K)$ to the right side one recovers the identity!