1. February 19

Recall that last time we showed that

Corollary 1.1. *Fix $q \in L^\infty(\Omega)$. There exists $\tau > 0$ such that for all $u \in C_0^2(\Omega)$,*

$$\tau \|u\|_{L^2(\Omega)} \lesssim \|(\triangle_{\pm\tau} + q)u\|_{L^2(\Omega)}.$$

By a Hahn-Banach argument, we get the following result.

Corollary 1.2. *Suppose $f \in L^2(\Omega)$, $q \in L^\infty(\Omega)$. Then for sufficiently large τ there exists $u \in L^2(\Omega)$ such that*

$$(\triangle_\tau + q)u = f$$

and

$$\|u\|_{L^2(\Omega)} \lesssim \tau^{-1}\|f\|_{L^2(\Omega)}.$$

This is what we need to prove the existence of CGOs.

1.1. **CGO Solutions and the Inverse Problem.**

Proposition 1.3. *Suppose $q \in L^\infty(\Omega)$. Then for sufficiently large τ, there exists a solution of the form*

$$u = e^{\tau x_1}(e^{i\tau x_2} + r)$$

to the equation

$$(\triangle + q)u = 0$$

with

$$\|r\|_{L^2(\Omega)} \leq \tau^{-1}\|q\|_{L^\infty(\Omega)}$$

Proof. By Corollary 1.2, there exists, for sufficiently large τ, a solution r to the equation

$$(\triangle + q)r = -qe^{i\tau x_2}$$

with

$$\|r\|_{L^2(\Omega)} \lesssim \tau^{-1}\|q\|_{L^2(\Omega)} \lesssim \tau^{-1}\|q\|_{L^\infty(\Omega)}.$$

Now one can check that

$$u = e^{\tau x_1}(e^{i\tau x_2} + r)$$

solves

$$(\triangle + q)u = 0$$

as desired. $\square$

By changing coordinates, we could write this in a number of other ways – for instance, we could write

$$u = e^{\tau x_1}(e^{i\tau(ax_2 + bx_3)} + r)$$

as long as $a^2 + b^2 = 1$.

While we're here, let's finish the proof of identifiability in the inverse problem:

Theorem 1.4. *Suppose $q_1, q_2 \in L^\infty(\Omega)$, and $\Lambda_{q_1} = \Lambda_{q_2}$. Then $q_1 = q_2$.*

Proof. By the integration by parts argument from February 12, we know that if u_1, u_2 solve

$$(\triangle + q_1)u_1 = (\triangle + q_2)u_2 = 0$$

on Ω, then $\Lambda_{q_1} = \Lambda_{q_2}$ implies that

$$\int_\Omega (q_2 - q_1)u_1 u_2 dx = 0. \tag{1.1}$$

Now by Proposition 1.3, we can take

$$u_1 = e^{\tau x_1}(e^{i\tau(ax_2+bx_3)} + r_1)$$

for sufficiently large τ, where $a^2 + b^2 = 1$. By changing coordinates, the same argument also tells us we can take

$$u_2 = e^{-\tau x_1}(e^{i\tau(-ax_2+bx_3)} + r_2).$$

Plugging these into the integral identity (1.1) gives

$$\int_\Omega (q_2 - q_1)e^{i\tau bx_3}(1 + r_1 + r_2 + r_1 r_2)dx = 0.$$

Now set $b = \beta\tau^{-1}$, so

$$\int_\Omega (q_2 - q_1)e^{i\beta x_3}(1 + r_1 + r_2 + r_1 r_2)dx = 0.$$

If we take $\tau \to \infty$, we get

$$\int_\Omega (q_2 - q_1)e^{i\beta x_3} dx = 0.$$

We can do this for any choice of β and x_3, which shows that the Fourier transform of $q_2 - q_1$ is zero. This shows that $q_2 = q_1$, so we're done.

$\square$

1.2. **Proof of the Carleman Estimate.** Recall that our Carleman estimate was the following.

Theorem 1.5. *For all $\tau > 0$ and $u \in C_0^2(\Omega)$,*

$$\tau\|u\|_{L^2(\Omega)} \lesssim \|(\triangle_{\pm\tau} + q)u\|_{L^2(\Omega)}.$$

Proof of Theorem 1.5. Suppose $\tau > 0$ and $u \in C_0^2(\Omega)$. The expression

$$\|\triangle_\tau u\|^2_{L^2(\Omega)} = (\triangle_\tau u, \triangle_\tau u).$$

is practically crying out to be integrated by parts. Explicitly, we have

$$\triangle_\tau u = (\triangle + 2\tau\partial_1 + \tau^2)u.$$

Notice that the terms of $\triangle_\tau$ will act very differently under integration by parts: $\triangle$ and τ^2 are self adjoint (not even any boundary terms, since $u \in C_0^2(\Omega)$), but $2\tau\partial_1$ is not.) Let's set

$$A = \triangle + \tau^2$$

and

$$B = 2\tau\partial_1.$$

Then

$$\begin{aligned}
\|\triangle_\tau u\|^2_{L^2(\Omega)} &= ((A+B)u,(A+B)u) \\
&= (Au,Au)+(Au,Bu)+(Bu,Au)+(Bu,Bu) \\
&= \|Au\|^2_{L^2(\Omega)} + ((AB-BA)u,u) + \|Bu\|^2_{L^2(\Omega)}
\end{aligned}$$

The integration by parts leaves no boundary terms, since we assumed that $u \in C_0^2(\Omega)$.

In general, proving a Carleman estimate now requires showing that the commutator is positive or at least not too negative. But we have a simple estimate to prove and we have the simplest possible commutator: $AB - BA = 0$. Therefore

$$\|\triangle_\tau u\|^2_{L^2(\Omega)} = \|Au\|^2_{L^2(\Omega)} + \|Bu\|^2_{L^2(\Omega)}.$$

A Poincaré inequality tells us that

$$\|Bu\|_{L^2(\Omega)} = 2\tau\|\partial_1 u\|_{L^2(\Omega)} \gtrsim \tau\|u\|_{L^2(\Omega)},$$

so

$$\|\triangle_\tau u\|^2_{L^2(\Omega)} \gtrsim \tau^2\|u\|^2_{L^2(\Omega)},$$

and the result follows.

□

For completeness, we can prove the Poincaré inequality as well: if $u \in C_0^2(\Omega)$, then u has a C^2 extension by zero to all of $\mathbb{R}^n$, and

$$u(x) = \int_{-\infty}^{x_1} \partial_1 u(t,x')dt.$$

By Jensen's inequality,

$$|u(x)| \lesssim \left(\int_{-\infty}^{x_1} |\partial_1 u(t,x')|^2 dt\right)^{\frac{1}{2}}.$$

Integrating, we get

$$\|u\|^2_{L^2(\mathbb{R}^n)} \lesssim \int_{\mathbb{R}^{n-1}} \int_{-C}^{C} \int_{-\infty}^{x_1} |\partial_1 u(t,x')|^2 \, dt \, dx_1 \, dx'.$$

Here the integral in x_1 is on a bounded interval because u is compactly supported. Fubini's Theorem tells us that

$$\|u\|^2_{L^2(\mathbb{R}^n)} \lesssim \int_{-C}^{C} \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} |\partial_1 u(t,x')|^2 \, dt \, dx' \, dx_1.$$

In other words,

$$\|u\|^2_{L^2(\mathbb{R}^n)} \lesssim 2C\|\partial_1 u(t,x')\|^2_{L^2(\mathbb{R}^n)}.$$

1.3. **A Direct Inverse.** There are many ways to obtain a direct inverse. The slickest is probably to use Fourier series in place of the Fourier transform. Let Q be the cube $[-\pi, \pi]^n$.

Theorem 1.6. *There exists an operator* $\triangle_\tau^{-1} : L^2(Q) \to L^2(Q)$ *such that*

$$\triangle_\tau \triangle_\tau^{-1} f = f$$

for all $f \in L^2(Q)$*, and*

$$\|\triangle_\tau^{-1} f\| \leq \tau^{-1} \|f\|_{L^2(Q)}.$$

If Ω is bounded, we can stick it inside a cube; WLOG the sube is Q. Consider the equation

$$\triangle_\tau u = (\triangle + 2\tau\partial_1 + \tau^2)u = f.$$

Instead of taking Fourier transforms, let's instead expand both sides in Fourier series:

$$u = \sum u_N e^{iN\cdot x}.$$

Matching like terms, we get

$$(-|N|^2 + 2i\tau N_1 + \tau^2)u_N = f_N.$$

We're still stuck with the problem that we're not allowed to divide by the coefficient of u_N to get u_N in terms of f_N. But that's just because we chose the wrong basis.

Lemma 1.7. *Fix* $H \in \mathbb{R}^n$*, and suppose* $f \in L^2(Q)$*. Then* f *has a unique representation*

$$f(x) = \sum_{N\in\mathbb{Z}^n} f_N e^{i(N+H)\cdot x}$$

where the sum converges in the L^2 *sense, and*

$$\|f\|_{L^2(Q)}^2 = \sum_{N\in\mathbb{Z}^n} |f_N|^2$$

Proof. Just take the Fourier series representation of $f(x)e^{-iH\cdot x}$ instead! $\square$

Note that when we expand $F = \partial_j f$ in terms of this Fourier series, we get

$$F_N = i(N_j + H_j)f_N.$$

Proof of Theorem 1.6. Consider the equation

$$\triangle_\tau u = (\triangle + 2\tau\partial_1 + \tau^2)u = f.$$

Choose $H = (\frac{1}{2}, \ldots, \frac{1}{2})$ and expand both sides in the Fourier series given by (1.7). Matching like terms gives us

$$(|N+H|^2 + 2i\tau(\frac{1}{2} + N_1) + \tau^2)u_N = f_N.$$

Note that

$$||N+H|^2 + 2i\tau(\frac{1}{2} + N_1) + \tau^2| \geq \tau$$

for all $N \in \mathbb{Z}^n$. Therefore we can divide through by it, and write

$$u_N = \frac{f_N}{(|N+H|^2 + 2i\tau(\frac{1}{2} + N_1) + \tau^2)} \tag{1.2}$$

with

$$|u_N| \leq \tau^{-1} f_N.$$

We can use this equation to define the operator $\triangle_\tau^{-1}$: for

$$f(x) = \sum_{N \in \mathbb{Z}^n} f_N e^{i(N+H)\cdot x}$$

$$\triangle_\tau^{-1} f = \sum_{N \in \mathbb{Z}^n} \frac{f_N e^{i(N+H)\cdot x}}{(|N+H|^2 + 2i\tau(\frac{1}{2} + N_1) + \tau^2)}.$$

Note that $\triangle_\tau^{-1}$ is indeed a left and right inverse to $\triangle_\tau$ and

$$\begin{aligned} \|\triangle_\tau^{-1} f\|^2_{L^2(Q)} &= \sum_{N \in \mathbb{Z}^n} \left| \frac{f_N}{(|N+H|^2 + 2i\tau(\frac{1}{2} + N_1) + \tau^2)} \right|^2 \\ &\leq \sum_{N \in \mathbb{Z}^n} \tau^{-2} |f_N|^2 \\ &= \tau^{-2} \|f\|^2_{L^2(Q)} \end{aligned}$$

□

This trick has its limitations – it doesn't work in unbounded domains and typically interacts badly with boundary values, unless you happen to be on a cube. On the other hand, in applications where you have compactly supported functions in a bounded domain, this can be great. You can prove the Poincaré inequality very easily with this trick, for instance.

It might be helpful to note that the Carleman estimate follows immediately from the existence of an inverse.

To see this, suppose $u \in C_0^2(\Omega)$. The existence of the inverse says that $u = \triangle_\tau^{-1} v$ for some $v \in L^2(\Omega)$, with

$$\|u\|_{L^2(\Omega)} \leq \tau^{-1} \|v\|_{L^2(\Omega)}.$$

But $u = \triangle_\tau^{-1} v$ implies that $v = \triangle_\tau u$, so

$$\|u\|_{L^2(\Omega)} \leq \tau^{-1} \|\triangle_\tau u\|_{L^2(\Omega)}.$$

2. February 23

2.1. **Reconstruction is hard.** Theorem 1.4 is an identifiability result: it shows that the map $q \mapsto \Lambda_q$ is one-to-one, but it doesn't give a formula for reconstructing q from Λ_q.

How difficult is it to give such a formula?

Recall that on February 12 we proved the general integration by parts formula

$$\int_\Omega (q_2 - q_1) u_1 u_2 \, dx = \int_{\partial\Omega} u_1 [\Lambda_{q_1} - \Lambda_{q_2}](u_2) \, dS$$

for u_j solving $(\triangle + q_j)u_j = 0$.

If we're just interested in recovering one q, we could write

$$\int_\Omega quv\,dx = \int_{\partial\Omega} u[\Lambda_q - \Lambda_0](v)\,dS$$

where $(\triangle + q)u = 0$ and $\triangle v = 0$. By using CGO solutions

$$\begin{aligned} u &= e^{\zeta_1\cdot x}(1+r) \\ v &= e^{\zeta_2\cdot x} \end{aligned}$$

we can arrange for

$$\lim_{\tau\to\infty}\int_\Omega quv\,dx = \int_\Omega qe^{i\xi\cdot x}\,dx = \hat{q}(\xi)$$

for any chosen ξ. Therefore

$$\hat{q}(\xi) = \lim_{\tau\to\infty}\int_{\partial\Omega} u[\Lambda_q - \Lambda_0](v)\,dS$$

for well chosen CGO solutions u and v. This looks like a reconstruction formula! There's only one problem: it's not clear from our construction of the CGO that we know what $u|_{\partial\Omega}$ is.

Everything else is known: Λ_q is known by hypothesis, Λ_0 depends only on the domain Ω, which is known, and v is just the harmonic function $e^{\zeta_2\cdot x}$ with ζ_2 chosen by us. It's only $u|_{\partial\Omega}$ which is unknown. In fact it's just $r|_{\partial\Omega}$ which is unknown.

One hope might be that $r|_{\partial\Omega}$ is small in the limit as $\tau\to\infty$, but there are no guarantees that this is the case. After all only the L^2 norm of r is small, and this gives no guarantees on the behaviour of r on a measure zero set.

So the problem of reconstruction is morally the problem of understanding the boundary behaviour of the CGO solution u, or at least its remainder term r.

2.2. **Finding $u|_{\partial\Omega}$.** Now r is given by an equation of the form

$$(\triangle_\tau + q)r = qe^{-i\tau x_2} \text{ on } \Omega.$$

or

$$(I + \triangle_\tau^{-1}q)r = qe^{-i\tau x_2} \text{ on } \Omega.$$

We need to find $r|_{\partial\Omega}$m but this is difficult, because the whole point is that we are ignorant of q.

On the other hand, when we defined $\triangle_\tau^{-1}$, we defined it on a bigger set than Ω: we defined it on a cube Q which contains Ω.

In the original Sylvester-Uhlmann paper, they actually managed to find an inverse on all of $\mathbb{R}^n$, by using weighted L^2 spaces.

Nachman realized that if you have such a thing, then you get a function r defined on all of $\mathbb{R}^n$, with

$$(\triangle_\tau + q)r = qe^{-i\tau x_2} \text{ on } \mathbb{R}^n.$$

In particular

$$\triangle_\tau r = 0$$

on $\Omega' = \mathbb{R}^n \setminus \Omega$. Equivalently,

$$\triangle e^{\tau x_1} r = 0 \text{ on } \Omega'.$$

Therefore $e^{\tau x_1} r$ is a harmonic function on Ω', and it's at least plausible that since you know the operator $\triangle_\tau^{-1}$, you understand the behaviour of r at infinity. So you have a harmonic function on Ω' whose behaviour at infinity is somewhat understood, and now it's plausible that you can find the boundary value of r! This is the key idea; a slightly more precise sketch follows.

Out of personal laziness, I'll express this in terms of

$$\triangle_\zeta = e^{-\zeta\cdot x} \triangle e^{\zeta \cdot x}$$

instead of $\triangle_\tau$.

We'll need a black box, which is the existence of the global $\triangle_\zeta^{-1}$.

Theorem 2.1. *The operator $\triangle_\zeta$ has an inverse $\triangle_\zeta^{-1} : L^2_\delta(\mathbb{R}^n) \to L^2_{\delta-1}(\mathbb{R}^n)$ such that*

$$\|\triangle_\zeta^{-1} f\|_{L^2_{\delta-1}(\mathbb{R}^n)} \lesssim \tau^{-1} \|f\|_{L^2_\delta(\mathbb{R}^n)}$$

The heart of the reconstruction idea is the equivalence of four problems: an integral equation, a global differential equation, an exterior differential equation, and a boundary problem. I'll present this in terms of u instead of r, following a paper of Mikko Salo.

Theorem 2.2. *The following problems are equivalent:*

- *The integral equation*

$$u + e^{\zeta\cdot x} \triangle_\zeta^{-1} e^{-\zeta\cdot x}(qu) = e^{\zeta\cdot x} \text{ on } \mathbb{R}^n. \tag{2.1}$$

- *The global differential equation*

$$\begin{aligned} (\triangle - q)u &= 0 \text{ on } \mathbb{R}^n \\ u &= e^{\zeta\cdot x}(1 + r) \text{ with } r \in \triangle_\zeta^{-1} L^2_\delta(\mathbb{R}^n) \end{aligned}$$

- *The exterior differential equation*

$$\begin{aligned} \triangle u &= \text{ in } \Omega' \\ u &= \in H^2(\Omega' \cap B_R(0)) \text{ for any } R > 0 \\ \partial_\nu u &= \Lambda_q u \text{ on } \partial\Omega \end{aligned}$$

  with the Sommerfeld radiation condition

$$\lim_{R\to\infty} \int_{|y|=R} G_\zeta(x,y)\partial_\nu u(y) - u(y)\partial_{\nu(y)} G_\zeta(x,y) dS(y) = e^{\zeta\cdot x}$$

  for almost every $x \in \mathbb{R}^n$, where G_ζ is a Green's function for the Laplacian defined in terms of $\triangle_\zeta^{-1}$, and almost every $x \in \mathbb{R}^n$.

- *The boundary integral equation*
$$\frac{1}{2}u + \int_{\partial\Omega} G_\zeta(x,y)\partial_\nu u(y) - u(y)\partial_{\nu(y)} G_\zeta(x,y) dS(y) = e^{\zeta\cdot x}$$
on $\partial\Omega$.

That the integral equation and the global differential equation are equivalent is really just the fact that $\triangle_\zeta^{-1}$ is a genuine inverse, up to some symbol-pushing.

Now if u solves the global differential equation, then it's clear that it must be harmonic on Ω', since q is supported only in Ω. The regularity of u follows from basic facts about harmonic functions, and the fact that u solves the global differential equation imposes boundary values on u at $\partial\Omega$ and at infinity, which can be understood as the last two conditions in the exterior problem.

For the remainder of the equivalences we have to be a little clearer about what G_ζ is. One can check that
$$e^{\zeta\cdot x}\triangle_\zeta^{-1}e^{-\zeta\cdot x}$$
is a right inverse for the regular Laplacian on $\mathbb{R}^n$. But so is
$$f \mapsto \int_{\mathbb{R}^n} G(x,y)f(y)dy,$$
where $G(x,y)$ is the standard Green's function for the Laplacian. One can check the implication:
$$e^{\zeta\cdot x}\triangle_\zeta^{-1}e^{-\zeta\cdot x}$$
is of the form
$$\int_{\mathbb{R}^n} G_\zeta(x,y)f(y)dy$$
where G_ζ is the regular Green's function $G(x,y)$ plus a harmonic function $H(x,y)$.

Then an integration by parts in the Sommerfeld radiation condition can be used to recover the integral equation.

Finally, an integration by parts on the exterior domain Ω', with the Sommerfeld radiation condition, gives the boundary problem.