1. February 26

1.1. **Hybrid Problems.** As we've seen above, tomography with elliptic equations is possible but difficult. This is a serious state of affairs because many forms of energy behave according to elliptic equations.

Other tomographic methods exist (e.g. X-ray tomography, ultrasound) but they often have drawbacks of their own. One of the most common drawbacks is that there may not be sufficient contrast between objects we're trying to identify. For example, if all non-bone tissue in the body absorbs X-rays at about the same rate (this is only sort of true but let's pretend here), X-ray tomography will not be super helpful in diagnosing soft-tissue problems.

Recently there has been some interest in using interactions between different types of physical phenomena to improve the state of affairs with elliptic inverse problems. These are sometimes called hybrid problems, because they act as a hybrid of two problems

We're going to talk about two of these: photoacoustic (sometimes thermoacoustic) tomography, and acousto-optic tomography.

1.2. **Photoacoustic Tomography.** The basic idea of photoacoustic tomography is as follows. We illuminate an object $\Omega \subset \mathbb{R}^3$ by some kind of radiation (usually microwaves – at low amplitudes this is a very safe thing to do.)

The microwaves are absorbed into Ω at differing rates, which causes them to heat up and expand. The expansion triggers a pressure wave through Ω, which can be measured at the boundary. From the boundary measurements of the pressure wave, we want to reconstruct the absorption coefficient.

Mathematically we can set up a simple version of the problem as follows.

Let u be the microwave intensity, and suppose u satisfies an elliptic equation like

$$\triangle u - \sigma u = 0 \text{ in } \Omega$$

with boundary condition $u|_{\partial\Omega} = g$ specified by us. Roughly speaking, this indicates that u diffuses isotropically throughout the medium with absorption governed by σ. More generally, we could replace the Laplacian with $\nabla \cdot \gamma \nabla$, but let's keep things simple for now.

The expansion of the medium creates a pressure proportional to the amount of microwave radiation absorbed, which is to say that it's proportional to σu. The proportion is governed by something called the Grüneisen coefficient, so the initial pressure is given by

$$f(x) = \Gamma(x)\sigma(x)u(x)$$

where Γ is the Grüneisen coefficient. Again, to keep things simple, let's assume that the Grüneisen coefficient is known.

The pressure wave then propagates according to a wave equation

$$\begin{aligned} \partial_t^2 p &= \triangle p \text{ in } \mathbb{R}^3 \times [0,\infty) \\ p(x,0) &= f(x) \\ \partial_t p(x,0) &= 0 \end{aligned}$$

Again the Laplacian should in general be replaced with some $c(x)\triangle$, but we'll keep things simple for now. Note that the pressure distorts the medium slightly, but not on a scale that we care about for reconstruction.

The measurement we take is $p(x,t)|_{\partial\Omega\times[0,\infty)}$. If you like you can think of this as a boundary value map defined by σ, of the form $\Lambda_\sigma : g \mapsto p(x,t)|_{\partial\Omega\times[0,\infty)}$. However, in this simple version of the problem, we will see that it suffices to use one fixed g.

Then the inverse problem is to recover σ from knowledge of $p(x,t)|_{\partial\Omega\times[0,\infty)}$. Note that since $p(x,t)$ is measured on a three dimensional subset (two space dimensions and one time dimension) there is substantially more information available here than there would be for a purely elliptic problem.

1.3. **Internal Functional.** Recovery in a hybrid problem typically proceeds in two steps. First, we use the extra boundary information to recover an internal functional: a function defined on all of Ω, and not just the boundary. Then we use this internal functional to recover the desired coefficient(s).

In photoacoustic tomography, this manifests itself in the following two step problem.

- First, we want to recover $f(x) = \Gamma(x)\sigma(x)u(x)$, the initial condition for the pressure wave, from the boundary data $p(x,t)|_{\partial\Omega\times[0,\infty)}$.
- Secondly, we want to recover σ from f.

Let's do this in order. How can we recover f from $p(x,t)|_{\partial\Omega\times[0,\infty)}$?

Recall that Huygen's principle in $\mathbb{R}^3$ tells us that a solution to

$$\begin{aligned} \partial_t^2 p &= \triangle p \text{ in } \mathbb{R}^3 \times [0,\infty) \\ p(x,0) &= f(x) \\ \partial_t p(x,0) &= 0 \end{aligned}$$

with f compactly supported inside Ω is eventually 0 inside a ball of radius R. In other words there exists some finite time T such that

$$p(x,t) = 0 \text{ for all } x \in \Omega, t > T - 1.$$

Why is this is useful? It tells us that all of the information originally inside Ω has flowed out through the boundary. We ought to be able to use the boundary information to reconstruct the initial condition f, and in fact we can, because of a second nice property of the wave equation: it's time reversible!

Consider the wave equation

$$\begin{aligned} \partial_t^2 q &= \triangle q \text{ in } \Omega \times [0,T] \\ q(x,0) &= 0 \\ \partial_t q(x,0) &= 0 \\ q(x,t)|_{\partial\Omega\times[0,T]} &= p(x,T-t)|_{\partial\Omega\times[0,T]} \end{aligned}$$

On the one hand, we can certainly recover the function q, since we know $p(x,T-t)|_{\partial\Omega\times[0,T]}$, and we can solve wave equations.

On the other hand, one can check that $q(x,t) = p(x,T-t)$ is a solution, by the time reversibility of the wave equation, and by the uniqueness of solutions to the wave equation, it must be the only solution. Therefore we can find $p(x,t)$ for any $t \in [0,\infty)$, and in particular, we have

$$f(x) = q(x,T) = p(x,0).$$

We can summarize:

Theorem 1.1. *Suppose $f(x) \in C(\Omega)$ and*

$$\begin{aligned} \partial_t^2 p &= \triangle p \textit{ in } \mathbb{R}^3 \times [0,\infty) \\ p(x,0) &= f(x) \\ \partial_t p(x,0) &= 0. \end{aligned}$$

There exists $T > 0$ such that $f(x) = q(x,T)$, where q is the solution to

$$\begin{aligned} \partial_t^2 q &= \triangle q \textit{ in } \Omega \times [0,T] \\ q(x,0) &= 0 \\ \partial_t q(x,0) &= 0 \\ q(x,t)|_{\partial\Omega\times[0,T]} &= p(x,T-t)|_{\partial\Omega\times[0,T]}. \end{aligned}$$

2. February 28

2.1. **Recovery of σ.** Now we need to recover σ from f.

Recall that

$$f(x) = \Gamma(x)\sigma(x)u(x).$$

Assuming the Grüneiser coefficient $\Gamma(x)$ is known and non-zero, we can divide through by Γ to obtain

$$H(x) = \frac{f(x)}{\Gamma(x)} = \sigma(x)u(x).$$

Therefore the product $H(x) = \sigma(x)u(x)$ is known. Now recall that

$$\triangle u - \sigma u = 0 \text{ in } \Omega$$

and $u|_{\partial\Omega} = g$. If σu is known, we need only solve a Poisson equation

$$\triangle u = H$$

in Ω with $u|_{\partial\Omega} = g$ to recover u. If $u > 0$ then we can divide H through by u to recover σ.

How do we know that we can guarantee that $u > 0$? Suppose g is positive and u is nonpositive somewhere inside Ω. It follows from continuity of u that u must have a nonpositive local minimum inside Ω. Since $u \leq 0$ at the nonpositive minimum, it follows from the equation $\triangle u - \sigma u = 0$ that u is superharmonic at the minimum, which is a contradiction.

Note that the recovery of σ described above is a reconstruction: it gives us a procedure to reconstruct σ from the boundary data $p(x,t)|_{\partial\Omega\times[0,\infty)}$

2.2. **Even more complicated.** In theory we should be able to use the wealth of data available from photoacoustics to reconstruct even more coefficients. As best as I can tell, the following is not a proof of anything – there's a hole in the theory that I'll point out below – but it can be used in practice.

Let's consider a problem where the light intensity satisfies a more general equation

$$\nabla \cdot \gamma \nabla u - \sigma u = 0 \text{ in } \Omega$$

with positive boundary condition $u|_{\partial\Omega} = g$ selected by us, and assume γ and σ are unknown. As in the previous problem the microwave sets off a pressure wave $p(x,t)$ which propagates according to the wave equation

$$\begin{aligned} \partial_t^2 p &= \triangle p \text{ in } \mathbb{R}^3 \times [0,\infty) \\ p(x,0) &= f(x) \\ \partial_t p(x,0) &= 0, \end{aligned}$$

where

$$f(x) = \Gamma(x)\sigma(x)u(x),$$

with the Grüneiser coefficient Γ being known and nonzero. We want to recover γ and σ from measurements of $p(x,t)|_{\partial\Omega\times[0,\infty)}$.

As before, for any given boundary microwave condition g, we can use $p(x,t)|_{\partial\Omega\times[0,\infty)}$ to reconstruct the initial condition $f(x)$ for the pressure wave, which in turn gives us

$$H(x) = \sigma(x)u(x).$$

Because we want to reconstruct two coefficients, let's make measurements for two different positive boundary conditions g_1 and g_2. Then we recover two internal functionals

$$H_1(x) = \sigma(x)u_1(x)$$

and

$$H_2(x) = \sigma(x)u_2(x)$$

where each u_j solves $\nabla \cdot \gamma \nabla u_j - \sigma u_j = 0$ with the boundary condition $u_j = g_j$.

Roughly speaking we have two equations in two unknowns (this is only very roughly the case, since they're differential equations) and we should be able to recombine them to get something useful.

Multiply the equation for u_1 by u_2 and multiply the equation for u_2 by u_1. Subtracting the resulting equations gives

$$u_1\nabla\cdot\gamma\nabla u_2 - u_2\nabla\cdot\gamma\nabla u_1 = 0.$$

Note that σ has been eliminated. Now we want to take advantage of the fact that the quotient

$$\frac{u_1}{u_2} = \frac{H_1}{H_2}$$

is known. (Recall that if g_2 is positive, division by u_2 is legal.) Observe that

$$\begin{aligned}\nabla\cdot\left(\gamma u_1^2\nabla\frac{u_2}{u_1}\right) &= \nabla\cdot\left(\gamma u_1^2\left(\frac{\nabla u_2}{u_1} - \frac{u_2}{u_1}\frac{\nabla u_1}{u_1}\right)\right)\\ &= \nabla\cdot(\gamma u_1\nabla u_2 - \gamma u_2\nabla u_1)\\ &= u_1\nabla\cdot\gamma\nabla u_2 - u_2\nabla\cdot\gamma\nabla u_1.\end{aligned}$$

(How did anyone discover this? I think this is where I'm supposed to provide insight but I've got nothing. I came upon a similar looking problem with a coauthor once and we just mashed the equations around on a board until we discovered something useful. If there's a better way no one told me what it was.) Therefore

$$\nabla\cdot\left(\gamma u_1^2\nabla\frac{u_2}{u_1}\right) = 0,$$

which means that

$$\nabla\cdot\left(\gamma u_1^2\nabla\frac{H_2}{H_1}\right) = 0.$$

The quantity H_2/H_1 is known, so we can read this as a transport equation for the unknown quantity γu_1^2. Assuming we can measure γu_1^2 at the boundary, and H_2/H_1 has a decent gradient field, we can recover γu_1^2 inside Ω. This is a big assumption! As far as I know there's no theory that guarantees that the gradient field of H_2/H_1 allows one to solve the transport equation throughout Ω. In practice, though, since H_2/H_1 is known, one would be able to see exactly where the reconstruction fails if it fails, and even pick different H_2/H_1 if necessary.

Does the quantity γu_1^2 look familiar? It's the square of the quantity $v = \sqrt{\gamma}u_1$, which is the Liouville change of variables we made when we were looking at Calderón's problem!

We have

$$\triangle v + qv - \frac{\sigma}{\gamma}v = 0$$

where

$$q = -\frac{\triangle\sqrt{\gamma}}{\sqrt{\gamma}}.$$

Since v is known, it follows that

$$a = q - \frac{\sigma}{\gamma}$$

is known. Moreover

$$H_1 = \sigma u_1 = \frac{\sigma}{\sqrt{\gamma}} v$$

is known, so

$$b = \frac{\sigma}{\sqrt{\gamma}}$$

is known. Finally we can rewrite a in terms of b as

$$a = -\frac{\triangle\sqrt{\gamma}}{\sqrt{\gamma}} - \frac{b}{\sqrt{\gamma}}.$$

Viewing this as an equation in $\sqrt{\gamma}$, we get

$$-\triangle\sqrt{\gamma} = a\sqrt{\gamma} + b.$$

Since a and b are known, and $\sqrt{\gamma}$ on the boundary can be measured, we can reconstruct $\sqrt{\gamma}$ and use it in the equation for b to reconstruct σ also.