1. February 5

1.1. Independence of Angle. Most of this is based on a partially-buttocked reading of the paper "Transport Equations for Elastic and Other Waves in Random Media" by Leonid Ryzhik, George Papanicolaou, and Joseph Keller.

Let's consider an RTE in two dimensions

$$\theta \cdot \nabla_x u(x,\theta) = -\sigma(x)u(x,\theta) + \int_{S^{n-1}} k(x,\theta,\theta')u(x,\theta')\,d\theta'. \tag{1.1}$$

We'll assume slightly stronger conditions in two respects than we normally do. First, let's suppose σ and k are positive and smooth.

Then we'll suppose that k is very isotropic:

$$k(x,\theta,\theta') = K(x,\theta\cdot\theta')$$

for some function $K : X \times [-1,1]$. This is another way of saying the strength of scattering between θ and θ' depends only on the angle between them.

On the other hand, we'll assume that the dominance of absorption over scattering breaks down:

$$\int [k(x,\theta,\theta') - \sigma(x)]d\theta = 0. \tag{1.2}$$

Ok. Now let's suppose $\tilde{u}(x,\theta)$ is a nonnegative solution to (1.1), let $\varepsilon > 0$ be small, and let's set

$$u(x,\theta) = \tilde{u}(x/\varepsilon,\theta).$$

Let's think about this: u is like a zoomed out version of $\tilde{u}$: what happens on very large scales with $\tilde{u}$ happens at normal scales with u. This makes sense if we're going to think about diffusion: it's a macro approximation of what's happening at small scales.

So what does u do? Plugging into (1.1), we see that

$$\varepsilon\theta \cdot \nabla_x u(x,\theta) = -\sigma(x)u(x,\theta) + \int_{S^{n-1}} k(x,\theta,\theta')u(x,\theta')\,d\theta'. \tag{1.3}$$

Now let's be physicists: we'll assume that u takes the form

$$u(x,\theta) = u_0(x,\theta) + \varepsilon u_1(x,\theta) + \varepsilon^2 u_2(x,\theta) + \dots$$

Fix x. Substituting this expansion into (1.3), and matching like powers of ε, we see that

$$-\sigma(x)u_0(x,\theta) + \int_{S^{n-1}} k(x,\theta,\theta')u_0(x,\theta')\,d\theta' = 0. \tag{1.4}$$

For a fixed x, this is saying that $u_0(x,\cdot)$ is a non-negative eigenfunction for the operator

$$A_2(u)(\theta) = \int_{S^{n-1}} k(x,\theta,\theta')u_0(x,\theta')\,d\theta'.$$

This is a positive symmetric operator since k is positive, so the spirits of functional analysis assure us that it has exactly one non-negative eigenfunction, and this must be $u_0(x,\cdot)$. Moreover, this function must be independent of angle: if it weren't, then $u_0(x,\theta+\theta_0)$ would also be a nonnegative eigenvalue by a change of variables.

So the principal term in the expansion is independent of angle! This is what we have to have to do a diffusion approximation: we have to be able to drop out the angular variable. So far so good. Now we need to derive a governing equation for $u_0(x)$.

There are two ways to proceed:

1.2. **Ryzhik-Papanicolaou-Keller.** So let's look at the $O(\varepsilon)$ term:

$$\theta \cdot \nabla u_0(x) = \int k(x,\theta,\theta')u_1(x,\theta')d\theta' - \sigma(x)u_1(x,\theta). \tag{1.5}$$

We argue that $\theta \cdot \nabla u_0$ is also an eigenfunction for A_2: if we pick coordinates such that $\theta = \hat{x}_1$, then

$$\int k(x,\theta,\theta')\theta' \cdot \nabla u_0(x)d\theta' = \int K(x,\theta_1')\theta' \cdot \nabla u_0(x)d\theta'.$$

How in practice can you integrate over θ' in two dimensions? One way is to parametrize $\theta = (\cos(t), \sin(t))$ and take t from 0 to 2π:

$$\int k(x,\theta,\theta')\theta' \cdot \nabla u_0(x)d\theta' = \int_0^{2\pi} K(x,\cos(t))(\cos(t)\partial_1 u_0(x) + \sin(t)\partial_2 u_0(x))dt$$

But now the second term of the integrand is odd so we get

$$\int k(x,\theta,\theta')\theta' \cdot \nabla u_0(x)d\theta' = A\int_{-1}^{1} K(x,\theta_1')\theta_1'\partial_1 u_0(x)d\theta_1'$$

for some constant A. Now the $\partial_1 u_0(x)$ lifts neatly out of the integral:

$$\int k(x,\theta,\theta')\theta' \cdot \nabla u_0(x)d\theta' = A\int_{-1}^{1} K(x,\theta_1')\theta_1'd\theta_1'\partial_1 u_0(x).$$

But $\partial_1 u_0(x) = \theta \cdot \nabla u_0(x)$ because of the way we chose coordinates, so

$$\int k(x,\theta,\theta')\theta' \cdot \nabla u_0(x)d\theta' = A\int_{-1}^{1} K(x,\theta_1')\theta_1'd\theta_1'\theta \cdot \nabla u_0(x).$$

This shows that $\theta \cdot \nabla u_0$ is an eigenfunction for A_2. Then (1.5) says that

$$A_2 u_1 - \sigma u_1 = v$$

where v is an eigenvector of A_2. If you write $u_1 = av + bv'$, you see that $A_2 u_1 = a\lambda v + bA_2 v' = v + a\sigma v + b\sigma v'$, so v' must be u_0. Therefore

$$u_1 = a\theta \cdot \nabla u_0(x) + bu_0(x).$$

Note that by jamming the bu_0 term back into the first term in the approximation, we may as well write

$$u_1(x,\theta) = a(x)\theta \cdot \nabla u_0(x).$$

Sadly, substituting this back into (1.5) doesn't yield anything useful; it just lets us calculate a.

This isn't yet a useful governing equation for u_0, because there's a u_1 term in it. So let's move on and look at the $O(\varepsilon^2)$ term in the equation. We get

$$\theta \cdot \nabla u_1 = A_2 u_2 - \sigma u_2. \tag{1.6}$$

We can sub in $u_1 = a\theta \cdot \nabla u_0$ to get

$$\theta \cdot \nabla[a(x)\theta \cdot \nabla u_0](x) = A_2 u_2(x,\theta) - \sigma u_2(x,\theta).$$

The left side is beginning to look like a diffusion equation but the left side doesn't have anything in terms of u_0. This is a pain, but there is a solution: we can integrate both sides in θ. The condition (1.2) tells us that the left side vanishes when we do this, and we're left with

$$\int_{S^{n-1}} \theta \cdot \nabla[a(x)\theta \cdot \nabla u_0](x)\, d\theta = 0.$$

Now let's parametrize $\theta = (\cos(t), \sin(t))$ like we did before, so we get

$$\int_0^{2\pi} (\cos^2(t)\partial_1(a(x)\partial_1 u_0) + \cos(t)\sin(t)(\partial_1(a(x)\partial_2 u_0) + \partial_2(a(x)\partial_1 u_0)) + \sin^2(t)\partial_2(a(x)\partial_2 u_0))\, dt = 0.$$

The middle terms are odd in the t variable, so what's left is

$$\int_0^{2\pi} \cos^2(t) dt \partial_1(a(x)\partial_1 u_0) + \int_0^{2\pi} \sin^2(t) dt \partial_2(a(x)\partial_2 u_0) = 0,$$

which is precisely

$$\nabla \cdot a\nabla u_0 = 0.$$