1. February 7

1.1. **Diffusion modeled optical tomography.**

So let's look at an elliptic inverse problem.

Now we have an equation for the light intensity $u : X \to \mathbb{R}$ of the form

$$\nabla \cdot \gamma \nabla u = 0 \text{ in } X.$$

We are measuring the boundary values of u on ∂X and want to recover γ. Let's pause a moment and decide what boundary values of u we can measure.

Certainly we should be able to measure the light intensity on the boundary. That is, we should be able to measure $u|_{\partial X}$. But we expect from the general theory of elliptic equations that any sufficiently regular $u|_{\partial X}$ should give rise to a solution to the equation above.

So we need to be able to measure an additional quantity on the boundary. The obvious candidate is $\partial_\nu u|_{\partial X}$: the other natural boundary condition for an elliptic equation. Is this reasonable to measure? Note that the derivation of the diffusion equation tells us that $\gamma(x)\nabla u(x)$ represents the flow at x – it's this flow which is divergenceless.

So a measurement of $\gamma \nabla u$ at the boundary is really a measure of the flow of light particles there. Assuming we know γ at the boundary (and why not; we're at the boundary after all), the tangential components of $\gamma \nabla u$ can be recovered from $u|_{\partial X}$, so we're really only interested in measuring $\nu \cdot \gamma \nabla u = \gamma \partial_\nu u|_{\partial X}$.

Now we can set these measurements up in terms of a boundary value map:

If we impose the intensity $u|_{\partial X} = f$ on the boundary, then inside X the intensity u must solve the boundary value problem

$$\begin{aligned} \nabla \cdot \gamma \nabla u &= 0 \\ u|_{\partial X} &= f. \end{aligned}$$

Then we can measure $\gamma \partial_\nu u$ at the boundary, so we have a boundary value map $\Lambda_\gamma : f \mapsto \gamma \partial_\nu u$. The inverse problem is to determine γ from Λ_γ. Of course this is just Calderón's problem.

1.2. **Calderón's Problem.**

Recall that in Calderón's problem, we have a bounded smooth domain $\Omega \subset \mathbb{R}^3$. We let γ be a positive function on Ω, representing the electrical conductivity. If we impose the electrical potential $u|_{\partial \Omega} = f$ on the boundary, then inside Ω the potential u must solve the boundary value problem

$$\begin{aligned} \nabla \cdot \gamma \nabla u &= 0 \\ u|_{\partial \Omega} &= f, \end{aligned} \tag{1.1}$$

so we get a boundary value map $\Lambda_\gamma : f \mapsto \gamma \partial_\nu u$, where $\gamma \partial_\nu$ represents the boundary current flux. We want to know if knowledge of Λ_γ determines γ.

Let's codify our assumptions a little bit better. For starters, let's assume that γ is smooth. There are a number of spaces that we could require f to live in, but it's traditional to assume that $f \in H^{\frac{1}{2}}(\partial \Omega)$. Then some elliptic theory tells us that there exists a unique solution $u \in H^1(\Omega)$ (in the weak sense) to (1.1), and $\partial_\nu u \in H^{-\frac{1}{2}}(\partial\Omega)$. Therefore

$$\Lambda_\gamma : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega).$$

If these spaces bother you it's ok to assume that everything is actually smooth.

How do we make this problem tractable? Clearly we have to integrate something by parts. Suppose v is a reasonable function – smooth, say. Consider

$$\int_\Omega \nabla \cdot \gamma \nabla u \, v \, dx.$$

If u solves (1.1) then

$$0 = \int_\Omega \nabla \cdot \gamma \nabla u \, v \, dx.$$

Integrating by parts gives

$$0 = -\int_\Omega \gamma \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \gamma \partial_\nu u \, v \, dS.$$

Aha! The $\gamma\partial_\nu u$ is actually just $\Lambda_\gamma f$, so we have

$$\int_\Omega \gamma \nabla u \cdot \nabla v \, dx = \int_{\partial\Omega} \Lambda_\gamma(f) \, v \, dS. \tag{1.2}$$

This tells us that knowing $\Lambda_\gamma f$ means knowing the interior integral on the left side: our integration by parts has propagated our boundary knowledge into the interior.

1.3. **Identifiability Problem.** We can leverage this to focus on the identifiability problem. Let's suppose γ_1 and γ_2 are positive smooth functions on Ω, with corresponding boundary value maps Λ_1 and Λ_2. We will assume that Λ_1 and Λ_2 the same, and try to prove that $\gamma_1 = \gamma_2$.

Let's fix a boundary function $f \in H^{\frac{1}{2}}(\partial\Omega)$ and suppose u_1 and u_2 solve (1.1) with $\gamma = \gamma_1, \gamma_2$ respectively.

The integration by parts identity (1.2) tells us that

$$\int_\Omega \gamma_1 \nabla u_1 \cdot \nabla u_2 \, dx = \int_{\partial\Omega} \Lambda_1(f) \, f \, dS$$

and

$$\int_\Omega \gamma_2 \nabla u_2 \cdot \nabla u_1 \, dx = \int_{\partial\Omega} \Lambda_2(f) \, f \, dS.$$

Subtracting, we get

$$\int_\Omega (\gamma_1 - \gamma_2) \nabla u_1 \cdot \nabla u_2 \, dx = \int_{\partial\Omega} (\Lambda_1 - \Lambda_2)(f) \, f \, dS.$$

Now if $\Lambda_1 = \Lambda_2$, then

$$\int_\Omega (\gamma_1 - \gamma_2) \nabla u_1 \cdot \nabla u_2 \, dx = 0. \tag{1.3}$$

Does this show that $\gamma_1 = \gamma_2$? It does if the set

$$\{\nabla u_1 \cdot \nabla u_2 | u_j \text{ solve (1.1) with } \gamma = \gamma_j\}$$

is large enough. Is it large enough?

1.4. **Calderón's Argument.** Calderón didn't solve this problem. (If you've developed the entire real theory of singular integrals from scratch, it turn out you can write papers in which you don't solve problems.)

But he did show something. Let's take a closer look at the elliptic equation (1.1). We can rewrite

$$0 = \nabla \cdot \gamma \nabla u = \gamma \triangle u + \nabla \gamma \cdot \nabla u.$$

Now γ is positive, so we can rewrite as

$$0 = \triangle u + \nabla(\log \gamma) \cdot \nabla u.$$

If $\nabla \log \gamma$ is small, then perhaps u is close to harmonic. So here's an easier, related question: is the set

$$\{\nabla u_1 \cdot \nabla u_2 | \triangle u_j = 0\}$$

large enough? Calderón proved the following proposition.

Proposition 1.1. *Suppose $\xi \in \mathbb{R}^3$. Then*

$$e^{-i\xi \cdot x} \in \{\nabla u_1 \cdot \nabla u_2 | \triangle u_j = 0\}.$$

Note that this really is a way of saying the set $\{\nabla u_1 \cdot \nabla u_2 | \triangle u_j = 0\}$ is large enough: if we could replace $\nabla u_1 \cdot \nabla u_2$ in (1.3) with any $e^{-i\xi \cdot x}$ then we would be able to say

$$\int_\Omega (\gamma_1 - \gamma_2) e^{-\xi \cdot x} \, dx = 0.$$

for any ξ. But this is another way of saying that the Fourier transform of $\gamma_1 - \gamma_2$ is zero, so $\gamma_1 - \gamma_2$ is zero as well.

Proof. Suppose $\zeta_1, \zeta_2 \in \mathbb{R}^3$ such that $\zeta_1 \cdot \zeta_2 = 0$ and $|\zeta_1| = |\zeta_2|$. LEt $\zeta = \zeta_1 + i\zeta_2$. Note that $e^{\zeta \cdot x}$ is harmonic, since

$$\triangle e^{\zeta \cdot x} = \zeta \cdot \zeta e^{\zeta \cdot x} = 0.$$

Moreover $e^{-\bar{\zeta} \cdot x}$ is harmonic as well, for the same reason. Now

$$\nabla e^{\zeta \cdot x} \cdot \nabla e^{-\bar{\zeta} \cdot x} = |\zeta|^2 e^{2i\zeta_2 \cdot x}.$$

The factor of $|\zeta|^2$ in front is annoying, but dividing by a constant doesn't change harmonicity, so let

$$u_1 = \frac{1}{|\zeta|} e^{\zeta \cdot x} \quad \text{and} \quad u_2 = \frac{1}{|\zeta|} e^{-\bar{\zeta} \cdot x},$$

and then

$$\nabla u_1 \cdot \nabla u_2 = e^{2i\zeta_2 \cdot x}.$$

By choosing $\zeta_2 = -\frac{1}{2}\xi$ and taking any ζ_1 such that $\zeta_1 \cdot \zeta_2 = 0$ and $|\zeta_1| = |\zeta_2|$, we're done. $\square$

This is promising stuff. A natural idea here is the following: since we're actually interested in solutions of

$$\triangle u + \nabla(\log \gamma) \cdot \nabla u = 0 \tag{1.4}$$

instead of solutions of

$$\triangle u = 0, \tag{1.5}$$

is it possible to say that solutions to the first equation are just small perturbations of solutions to the second?

2. February 12

2.1. Scaling and Tricks.

Why should we believe that solutions to (1.4) might be perturbations of solutions to (1.5)? If you think of things in terms of the Fourier transform, you can see right away that the first term of the operator $\triangle + \nabla \log \gamma \cdot \nabla$ dominates the second at large frequencies. This suggests that if we have a large frequency harmonic function, it's nearly a solution to (1.4).

Another way to get some intuition is to look at *scaling*. Suppose we set $\tilde{u}(x) = u(\tau x)$ for some large τ, and hit $\tilde{u}$ with operator $\triangle + \nabla \log \gamma \cdot \nabla$: we get

$$\tau^2[\triangle u](\tau x) + \tau \nabla(\log \gamma) \cdot [\nabla u](\tau x)$$

or

$$\tau^2 \widetilde{\triangle u} + \tau \nabla(\log \gamma) \cdot \widetilde{u}.$$

The first term dominates the second. If we had set u to be a harmonic function and taken $\tilde{u} = \tau^{-2} u(\tau x)$, then we see that we would have a solution up to $O(\tau^{-1})$.

We can make this domination stronger with a clever trick, which is useful to keep in mind whenever one faces a second order equation with a first order term.

Suppose u solves

$$\nabla \cdot \gamma \nabla u = 0.$$

Set $v = \gamma^{\frac{1}{2}} u$, so $u = \gamma^{-\frac{1}{2}} v$. Then

$$\nabla \cdot \gamma \nabla(\gamma^{-\frac{1}{2}} v) = 0.$$

In other words

$$\gamma \triangle(\gamma^{-\frac{1}{2}} v) + \nabla \gamma \cdot \nabla(\gamma^{-\frac{1}{2}} v) = 0.$$

Expanding, we get

$$\gamma^{\frac{1}{2}} \triangle v + 2\gamma \nabla \gamma^{-\frac{1}{2}} \cdot \nabla v + \gamma \triangle(\gamma^{-\frac{1}{2}}) v + \nabla \gamma \cdot \nabla \gamma^{-\frac{1}{2}} v + \gamma^{-\frac{1}{2}} \nabla \gamma \dot{\nabla} v = 0.$$

Note that

$$2\gamma \nabla \gamma^{-\frac{1}{2}} + \gamma^{-\frac{1}{2}} \nabla \gamma = 0,$$

so we get

$$\gamma^{\frac{1}{2}} \triangle v + (\gamma \triangle(\gamma^{-\frac{1}{2}}) + \nabla \gamma \cdot \nabla \gamma^{-\frac{1}{2}}) v = 0.$$

Dividing by $\gamma^{\frac{1}{2}}$ gives

$$\triangle v + (\gamma^{\frac{1}{2}} \triangle(\gamma^{-\frac{1}{2}}) + \gamma^{-\frac{1}{2}} \nabla \gamma \cdot \nabla \gamma^{-\frac{1}{2}}) v = 0.$$

Notice that we got rid of the first order term! We can write

$$(\triangle + q)v = 0$$

where

$$q = \gamma^{\frac{1}{2}}\triangle(\gamma^{-\frac{1}{2}}) + \gamma^{-\frac{1}{2}}\nabla\gamma\cdot\nabla\gamma^{-\frac{1}{2}} = -\gamma^{-\frac{1}{2}}\triangle(\gamma^{\frac{1}{2}}). \tag{2.1}$$

The entire argument can be reversed, so $(\triangle + q)v = 0$ if and only if $\nabla\cdot\gamma\nabla u = 0$. Now the scaling exercise shows that the first term is much stronger than the second: it's $O(\tau^2)$ bigger instead of just $O(\tau)$ bigger.

The price of this trick is that q is in general much less regular than γ: if $\gamma \in C^2$, for instance, then q is just continuous. As γ becomes less and less regular this trick becomes a worse and worse idea. Fortunately we decided early on that γ should be smooth, so this shouldn't bother us here.

2.2. **Schrödinger Inverse Problem.** The other price of this trick is that we have to recast the inverse problem in terms of q.

If we set $v = \gamma^{\frac{1}{2}}u$, then the above discussion tells us that $\nabla\cdot\gamma\nabla u = 0$ if and only if $(\triangle + q)v = 0$.

Let's suppose we know γ and $\partial_\nu\gamma$ on the boundary of Ω. Then if we know $u, \gamma\partial_\nu u|_{\partial\Omega}$ then we know $v, \partial_\nu v$ on the boundary of Ω, and vice versa.

Therefore knowing the map $\Lambda_\gamma : u \to \gamma\partial_\nu u$ is equivalent to knowing the map $\Lambda_q : v \to \partial_\nu v$ defined by the equation $(\triangle + q)v = 0$. This suggests that we should look at the following inverse problem:

Given $q \in C^\infty(\Omega)$, such that $q = \gamma^{\frac{1}{2}}\triangle\gamma^{\frac{1}{2}}$ for some positive smooth γ, the equation

$$(\triangle + q)v = 0$$

defines a boundary value map $\Lambda_q : v \mapsto \partial_\nu v$. Given Λ_q, can we determine q? This is sometimes called the Schrödinger inverse problem.

Suppose we can show that $\Lambda_{q_1} = \Lambda_{q_2}$ implies that $q_1 = q_2$. Then (2.1) says that $\Lambda_{q_1} = \Lambda_{q_2}$ implies that

$$\triangle\gamma_1^{\frac{1}{2}} + q_2\gamma_1^{\frac{1}{2}} = 0$$

In other words

$$\nabla(\gamma_2\cdot\nabla(\gamma_2^{-\frac{1}{2}}\gamma_1^{\frac{1}{2}})) = 0.$$

Now $\gamma_1^{\frac{1}{2}} = \gamma_2^{\frac{1}{2}}$ is a solution to this equation. But moreover we know this equation (viewed as an equation for $\gamma_1^{\frac{1}{2}}$) has a unique solution for any given boundary value. If we know that $\gamma_1^{\frac{1}{2}} = \gamma_2^{\frac{1}{2}}$ on the boundary of Ω, then it follows that $\gamma_1 = \gamma_2$. To sum up, we have shown that

- If $\Lambda_{q_1} = \Lambda_{q_2}$ implies that $q_1 = q_2$ in the Schrödinger inverse problem then $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ implies that $\gamma_1 = \gamma_2$ in the Calderón problem.

Therefore it suffices to show identifiability in the Schrödinger inverse problem.

To do this we need to do another integration by parts: Suppose $(\triangle + q_1)u_1 = 0$ and $(\triangle + q_2)u_2 = 0$. Define w so that $(\triangle + q_1)w = 0$ and $w = u_2$ on the boundary of Ω. Then integrating by parts, we get

$$\int_\Omega \triangle u_1(u_2 - w)dx = \int_\Omega u_1 \triangle(u_2 - w)dx + \int_{\partial\Omega} \partial_\nu(u_2 - w)dS - \int_{\partial\Omega} u_1 \partial_\nu(u_2 - w)dS.$$

The first boundary term vanishes since $u_2 = w$ on the boundary, and the second can be rewritten in terms of the Λ_{q_j}'s:

$$\int_\Omega \triangle u_1(u_2 - w)dx = \int_\Omega u_1 \triangle(u_2 - w)dx - \int_{\partial\Omega} u_1(\Lambda_{q_2}(u_2) - \Lambda_{q_1}(u_2))dS.$$

Meanwhile we can use the equations for u_1, u_2, and w to get

$$-\int_\Omega q_1 u_1(u_2 - w)dx = \int_\Omega u_1(-q_2 u_2 + q_1 w)dx - \int_{\partial\Omega} u_1(\Lambda_{q_2}(u_2) - \Lambda_{q_1}(u_2))dS.$$

Simplifying,

$$\int_\Omega (q_2 - q_1)u_1 u_2 dx = -\int_{\partial\Omega} u_1(\Lambda_{q_2}(u_2) - \Lambda_{q_1}(u_2))dS.$$

Therefore if $\Lambda_{q_1} = \Lambda_{q_2}$, then

$$\int_\Omega (q_2 - q_1)u_1 u_2 dx = 0. \tag{2.2}$$

Note that if we could take u_j of the form $e^{\zeta\cdot x}$ like we did in the Calderón problem, we'd be done, exactly as we were there.

3. February 14

3.1. **Inverses and Neumann Series.** Our actual challenge is now to create a solution to

$$(\triangle + q)u = 0 \tag{3.1}$$

that looks nearly like $u = e^{\zeta\cdot x}$. The scaling argument suggests that we actually want this to happen when $|\zeta|$ is very large. To keep things straight, let's suppose $\zeta = \hat{x}_1 + i\hat{x}_2$ and write $e^{\zeta\cdot x} = e^{\tau x_1 + ix_2}$, where $\tau \gg 1$ (we can always arrange this by a change of coordinates.)

Now to be more precise, we want to look for a solution to (3.1) of the form

$$u = e^{\tau x_1 + ix_2}(1 + R(x)),$$

where R is small for large τ.

This is mostly a matter of convention, but notice that the $e^{i\tau\zeta_2\cdot x}$ part of the exponential factor doesn't add anything to the size of anything, so let's write that we're looking for a solution to (3.1) of the form

$$u = e^{\tau x_1}(e^{i\tau x_2} + r_\tau(x)), \tag{3.2}$$

where r_τ is small when τ is large. (A moment's contemplation of (2.2) should convince you that we want r_τ to be small in the L^2 norm, since q_j are in L^∞.) For complicated (complex?) historical reasons solutions of the form indicated in (3.2) are known as complex geometrical optics solutions, or CGO solutions.

What do we need to prove to get a CGO? If we plug (3.2) into (3.1), we get

$$(\triangle + q)e^{\tau x_1}(e^{i\tau x_2} + r_\tau) = 0.$$

When the Laplacian hits $e^{\tau x_1}e^{i\tau x_2}$ we get zero, so

$$(\triangle + q)e^{\tau x_1}r_\tau = -qe^{\tau x_1}e^{i\tau x_2}.$$

Multiplying both sides by $e^{-\tau x_1}$, we get

$$(e^{-\tau x_1}\triangle e^{\tau x_1} + q)r_\tau = -qe^{i\tau x_2}.$$

You can see that the operator on the left side has the form of a second order operator with known coefficients plus an unknown perturbation. Let's write

$$e^{-\tau x_1}\triangle e^{\tau x_1} = \triangle_\tau,$$

so we have

(3.3) $$(\triangle_\tau + q)r_\tau = -qe^{i\tau x_2}.$$

This is the equation for r_τ. Now we have two choices. One is that we could show directly that $(\triangle_\tau + q)$ has an inverse, and the inverse is small. Then we'd have

$$r_\tau = -(\triangle_\tau + q)^{-1}qe^{i\tau x_2},$$

so r_τ is small. Another possibility is to invert $\triangle_\tau$ only: then we'd have

$$(I + \triangle_\tau^{-1}q)r_\tau = -\triangle_\tau^{-1}qe^{i\tau x_2}.$$

We can solve for r_τ via Neumann series if $\triangle_\tau^{-1}$ is small. Moreover if $\triangle_\tau^{-1}$ is really small as $\tau \to \infty$, then we'd get that r_τ is small for large τ also.

Therefore another way to proceed is just to show that $\triangle_\tau$ is invertible and the inverse is small.

Let's take a moment to decide whether or not this is hard to do. Explicitly $\triangle_\tau u = f$ means

$$(\triangle + 2\tau\partial_1 + \tau^2)u = f$$

If we take the Fourier transforms, we get

$$(-|\xi|^2 + 2i\tau\xi_1 + \tau^2)\hat{u} = \hat{f}$$

Now we begin to see the problem. If we had something like

$$(-|\xi|^2 + 2i\tau\xi_1 - \tau^2)\hat{u} = \hat{f}$$

then the fact that $(-|\xi|^2 + 2i\tau\xi_1 - \tau^2) \geq \tau^2$ would mean that we could just divide through by $-|\xi|^2 + 2i\tau\xi_1 - \tau^2$ to get

$$\hat{u} = \frac{\hat{f}}{-|\xi|^2 + 2i\tau\xi_1 - \tau^2}.$$

This gives us a formula for $\triangle_\tau^{-1}$, and Plancherel's Theorem would tell us that

$$\|\triangle_\tau^{-1}f\|_{L^2(\mathbb{R}^n)} \leq \tau^{-2}\|f\|_{L^2(\mathbb{R}^n)},$$

so $\triangle_\tau^{-1}$ is small. But we have no such luck! The polynomial $-|\xi|^2 + 2i\tau\xi_1 - \tau^2$ is zero when $\xi_1 = 0$ and $|\xi'| = \tau$. We can't divide through naively unless we can somehow be assured that $\hat{f}$ is zero on this circle, and there's no a priori reason to believe this. Adding q only makes things worse. So the conclusion is that this is going to be hard.

3.2. **Carleman Estimates and Duality.** One way out of this mess is to prove the following estimate instead.

Theorem 3.1. *Fix $\tau > 0$. Then for all $u \in C_0^2(\Omega)$,*

$$\tau \|u\|_{L^2(\Omega)} \lesssim \|\triangle_{\pm\tau} u\|_{L^2(\Omega)}.$$

This is sometimes called a Carleman estimate: it's a lower bound for an operator with a weight attached, and similar estimates show up in a huge number of PDE problems: inverse problems, control theory, unique continuation, wave dispersal, etc.

We'll get around to proving this eventually, but let's consider for a moment why this estimate is relevant. Essentially Theorem 3.1 is a lower bound on the $\triangle_\tau$ operator, which is linear. Therefore it says that $\triangle_\tau$ is one-to-one. This is almost the same as saying that $\triangle_\tau$ is invertible – if only it were onto.

Sadly $\triangle_\tau$ is probably not onto. We can fix this by making a choice of a right inverse – the fact that we can make this choice is guaranteed by the Hahn-Banach theorem.

Corollary 3.2. *Suppose $f \in L^2(\Omega)$. Then there exists $u \in L^2(\Omega)$ such that*

$$\triangle_\tau u = f$$

and

$$\|u\|_{L^2(\Omega)} \lesssim \tau^{-1} \|f\|_{L^2(\Omega)}.$$

Proof. Let $f \in L^2(\Omega)$ and consider the subspace of $L^2(\Omega)$ defined by

$$E = \{v \in L^2(\Omega) | \exists w \in C_0^2(\Omega) \triangle_{-\tau} w = v\}.$$

We can define a linear functional φ on E by

$$\varphi(v) = \int_\Omega wf = (w, f).$$

(check that this is actually a linear functional!). By Theorem 3.1,

$$|\varphi(v)| \le \|w\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)} \le \tau^{-1} \|v\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)},$$

so this is a bounded linear functional. By Hahn-Banach, it has an extension to the whole space $L^2(\Omega)$. Now what? The Riesz Representation Theorem says that there exists $u \in L^2(\Omega)$ such that $\|u\|_{L^2(\Omega)} \le \tau^{-1} \|v\|_{L^2(\Omega)}$ and

$$(w, f) = \varphi(v) = (v, u).$$

In other words,

$$(w, f) = (\triangle_{-\tau} w, u).$$

Integrating by parts,

$$(w, f) = (w, \triangle_\tau u).$$

This holds for every $w \in C_0^2(\Omega)$, so the conclusion is that $\triangle_\tau u = f$.

□

We can define G_τ to be the map that takes f to u. Note that G_τ is a right inverse to $\triangle_\tau$, but not a left inverse. That is to say, we can guarantee that $\triangle_\tau G_\tau f = f$, but not that $G_\tau \triangle_\tau u = u$.

Is the right inverse enough to work the Neumann series argument? It turns out that it is: if you set

$$r = (I - G_\tau q + G_\tau q G_\tau q - \ldots)(-G_\tau q e^{i\tau x_2})$$

then you see by applying $\triangle_\tau$ to both sides that r solves (3.3).

There's another neater way to proceed though: we can use the Carleman estimate to provide a right inverse directly for $(\triangle_\tau + q)$.

Corollary 3.3. *Fix $q \in L^\infty(\Omega)$. There exists $\tau > 0$ such that for all $u \in C_0^2(\Omega)$,*

$$\tau \|u\|_{L^2(\Omega)} \lesssim \|(\triangle_{\pm\tau} + q)u\|_{L^2(\Omega)}.$$

Proof. By basic L^p inequalities,

$$\|(\triangle_{\pm\tau} + q)u\|_{L^2(\Omega)} \le \|\triangle_{\pm\tau} u\|_{L^2(\Omega)} + \|q\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)}. \tag{3.4}$$

Theorem 3.1 guarantees us that there exists $C > 0$ such that

$$\tau \|u\|_{L^2(\Omega)} \le C \|\triangle_{\pm\tau} u\|_{L^2(\Omega)}$$

Choose $\tau > 2C\|q\|_{L^\infty(\Omega)}$. Then (3.4) says that

$$\|(\triangle_{\pm\tau} + q)u\|_{L^2(\Omega)} \le \|\triangle_{\pm\tau} u\|_{L^2(\Omega)} + \frac{1}{2}\tau \|u\|_{L^2(\Omega)},$$

so substituting this inequality into the Carleman estimate tells us that

$$\frac{1}{2}\tau \|u\|_{L^2(\Omega)} \le C \|(\triangle_{\pm\tau} + q)u\|_{L^2(\Omega)}.$$

□

By an argument very similar to the one in Corollary 3.2, we can prove one last corollary:

Corollary 3.4. *Suppose $f \in L^2(\Omega)$, $q \in L^\infty(\Omega)$. Then for sufficiently large τ there exists $u \in L^2(\Omega)$ such that*

$$(\triangle_\tau + q)u = f$$

and

$$\|u\|_{L^2(\Omega)} \lesssim \tau^{-1} \|f\|_{L^2(\Omega)}.$$

This corollary, together with the argument preceding equation (3.3), lets us prove that there are CGO solutions.

3.3. **CGO Solutions and the Inverse Problem.**

Proposition 3.5. *Suppose $q \in L^\infty(\Omega)$. Then for sufficiently large τ, there exists a solution of the form*

$$u = e^{\tau x_1}(e^{i\tau x_2} + r)$$

to the equation

$$(\triangle + q)u = 0$$

with

$$\|r\|_{L^2(\Omega)} \leq \tau^{-1}\|q\|_{L^\infty(\Omega)}$$

By changing coordinates, we could write this in a number of other ways – for instance, we could write

$$u = e^{\tau x_1}(e^{i\tau(ax_2+bx_3)} + r)$$

as long as $a^2 + b^2 = 1$.

While we're here, let's finish the proof of identifiability in the inverse problem:

Theorem 3.6. *Suppose $q_1, q_2 \in L^\infty(\Omega)$, and $\Lambda_{q_1} = \Lambda_{q_2}$. Then $q_1 = q_2$.*

Proof. By the integration by parts argument in Section 2.2, we know that if u_1, u_2 solve

$$(\triangle + q_1)u_1 = (\triangle + q_2)u_2 = 0$$

on Ω, then $\Lambda_{q_1} = \Lambda_{q_2}$ implies that

$$\int_\Omega (q_2 - q_1)u_1 u_2 dx = 0. \tag{3.5}$$

Now by Proposition 3.5, we can take

$$u_1 = e^{\tau x_1}(e^{i\tau(ax_2+bx_3)} + r_1)$$

for sufficiently large τ, where $a^2 + b^2 = 1$. By changing coordinates, the same argument also tells us we can take

$$u_2 = e^{-\tau x_1}(e^{i\tau(-ax_2+bx_3)} + r_2).$$

Plugging these into the integral identity (3.5) gives

$$\int_\Omega (q_2 - q_1)e^{i\tau bx_3}(1 + r_1 + r_2 + r_1 r_2)dx = 0.$$

Now set $b = \beta\tau^{-1}$, so

$$\int_\Omega (q_2 - q_1)e^{i\beta x_3}(1 + r_1 + r_2 + r_1 r_2)dx = 0.$$

If we take $\tau \to \infty$, we get

$$\int_\Omega (q_2 - q_1)e^{i\beta x_3}dx = 0.$$

We can do this for any choice of β and x_3, which shows that the Fourier transform of $q_2 - q_1$ is zero. This shows that $q_2 = q_1$, so we're done.

□