1. January 10

1.1. Introduction. In a typical PDE problem, we have a PDE with known coefficients, and we want to derive information about the solutions. An *inverse problem* is one that goes the other way around: we are given information about the solutions to a PDE, usually on the boundary of some domain, and we want to derive information about the coefficients. These problems arise in a number of practical contexts, from medical imaging to mineral exploration.

Having said that, the term "inverse problem" isn't really well defined. Instead of trying to define it, let's look at an example.

1.2. Calderón's problem. Consider the following physical problem: we want to determine the electrical conductivity inside of an object by making electrical measurements on the boundary. Mathematically the problem takes the following form.

Let Ω be a bounded domain in $\mathbb{R}^3$, and let γ be a positive function on Ω, representing the electrical conductivity. If we impose the electrical potential $u|_{\partial\Omega} = f$ on the boundary, then inside Ω the potential u must solve the boundary value problem

$$\begin{aligned} \nabla \cdot \gamma \nabla u &= 0 \\ u|_{\partial\Omega} &= f. \end{aligned}$$

For each solution u, we can record not only the boundary value $u|_{\partial\Omega} = f$ but also the boundary current flux $\partial_\nu u|_{\partial\Omega}$.

By imposing different boundary values f and measuring the resulting current flux, we can accumulate data determined by γ, and see if we can recover γ from this data.

One way to view this problem is to note that γ determines a boundary value map $\Lambda_\gamma : f \mapsto \partial_\nu u$, and ask if knowledge of this map determines γ.

This is a prototype for a large variety of inverse problems. We start with a physical imaging problem, which amounts mathematically to the following situation: we have a function u which satisfies some PDE on the interior of Ω. We are given a boundary value map generated by the PDE, and we want to recover the coefficients of that PDE.

In his original 1980 paper, Calderón actually asked five distinct questions, which provide a framework for inverse problems to this day.

(1) Uniqueness: If Λ_{γ_1} and Λ_{γ_2} are the same, does it follow that $\gamma_1 = \gamma_2$? In other words, is the map $\gamma \mapsto \Lambda_\gamma$ one-to-one?
(2) Reconstruction: Is there a formula or algorithm that gives γ in terms of Λ_γ?
(3) Stability: If Λ_{γ_1} and Λ_{γ_2} are close in some sense, does that mean that γ_1 and γ_2 are close in some sense? Note that if this is not true, then small errors in measurement can lead to huge errors in reconstruction.
(4) Range: What functions g can be represented as $\Lambda_\gamma(f)$? This knowledge has value for any practical reconstruction algorithm from real data.
(5) Numerics: Is there a numerical algorithm for reconstruction? How good is it?

Questions 4-5 are more or less outside the scope of this course, but we will investigate 1-3 for several systems.

Note that questions 1-3 are generally quite difficult problems! In most applications people settle for making pictures only without doing a full reconstruction.

1.3. **Optical Tomography.** To begin the course, we'll look at three problems which can all be understood as manifestations of an *optical tomography* problem.

In optical tomography, we send light into an object and look to see how much comes out at the boundary. From these measurements we want to understand the optical parameters of the inside of the object. In other words optical tomography is just looking, but in our case we want to create a full reconstruction, and this is not necessarily easy.

How difficult it is depends on the level of scattering – that is to say, how much the light bounces around inside the object and refuses to travel in straight lines.

Very high energy photons, like X-rays, typically have very low scattering, and to a first approximation the scattering can be ignored entirely. In medical applications, these high energy photons have the rather significant downside of giving you cancer, or at least an increased risk of cancer. Therefore one would like to use lower energy photons, but as the scattering increases reconstruction becomes progressively more difficult.

2. January 12

2.1. **X-Rays: The Albedo Operator.** Let's consider the case with no scattering first. In this case light travelling in a given direction θ continues to travel in the direction θ no matter what.

Let $X \subset \mathbb{R}^n$ represent the domain (bounded open set with smooth boundary) which we want to investigate. To simplify matters slightly we'll also assume X is convex; if X is not convex we can always take our measurements on the convex hull of X instead. (Actually, this sort of reasoning really implies that we could just take X to be a ball. If you like you can pretend WLOTMG in all subsequent discussion that X is just a ball.)

For each direction $\theta \in S^{n-1}$ we'll let $u_\theta : X \to \mathbb{R}$ represent the intensity of light in direction θ, so $u_\theta(x)$ is the amount of light travelling in direction θ at point x.

In the absence of scattering the only thing that governs light propagation is an absorption factor $\sigma : X \to \mathbb{R}$, (physically positive but mathematically who cares), and then the equation governing u_θ is

$$\theta \cdot \nabla u_\theta(x) = -\sigma(x) u_\theta(x).$$

Roughly speaking, this says that as we travel in the θ direction, u_θ decreases at a relative rate proportional to the absorption factor σ.

To state the boundary conditions we need to define the incoming and outgoing boundary conditions: we define

$$\Gamma_\pm^\theta = \{x \in \partial X | \nu(x) \cdot \theta \geq 0\}.$$

Roughly speaking, Γ_-^θ corresponds to the part of the boundary where light travelling in direction θ enters X, and Γ_+^θ corresponds to the part of the boundary where light travelling in direction θ is leaving X. (Draw a diagram! If X is genuinely a ball, what are $\Gamma_\pm^\theta$?) This gives rise to the alternate names: the incoming and outgoing boundaries, respectively.

As we'll see in a bit (see Theorem 2.1, if $\sigma \in C(X)$, the problem

$$\begin{aligned} \theta \cdot \nabla u_\theta(x) &= -\sigma(x) u_\theta(x) \text{ in } X \\ u_\theta(x)|_{\Gamma_-^\theta} &= f(x) \end{aligned} \tag{2.1}$$

has a unique solution $u_\theta \in C(X)$ for each $f \in C(\Gamma_-^\theta)$. (This isn't the optimal regularity but let's run with it.)

Now we define the albedo map $\mathcal{A}_\theta : C(\Gamma_-^\theta) \to C(\Gamma_+^\theta)$ by

$$\mathcal{A}_\theta(f) = u_\theta(x)|_{\Gamma_+^\theta},$$

where u_θ is the unique solution to (2.2).

This lets us state the inverse problem in its crudest form: if we know $\mathcal{A}_\theta$ for every θ, then can we reconstruct σ? (Note: it is not enough to consider just one θ: why?)

2.2. **X-Rays: The forward problem.** Let's pause a moment here to understand (2.2) in the ordinary PDE way: how do we know this is solvable and what do the solutions looks like?

The equation (2.2) is a simple transport equation and as such it's really an ODE in disguise. To see this, let's pick coordinates (x_1, x_*) on $\mathbb{R}^n$ so that x_1 is aligned with θ.

Then (2.2) becomes

$$\begin{aligned} \partial_1 u_\theta(x) &= -\sigma(x) u_\theta(x) \text{ in } X \\ u_\theta(x)|_{\Gamma_-^\theta} &= f(x) \end{aligned} \tag{2.2}$$

The first equation is an ODE we can solve by integrating factors: we get

$$u_\theta(x_1, x_*) = \exp\left(-\int_a^{x_1} \sigma(t, x_*)\, dt\right) \cdot C(x_*)$$

The contribution of the boundary condition can be understood as follows: the ray from x in the negative x_1 direction hits ∂X (since X is bounded) at exactly one point (since X is convex). Call this point x^-. This point x^- must be in Γ_-^θ, since travel in direction θ moves us inside X. Then the boundary condition tells us that $u_\theta(x^-) = f(x^-)$ so

$$u_\theta(x_1, x_*) = \exp\left(-\int_{x_1^-}^{x_1} \sigma(t, x_*)\, dt\right) \cdot f(x^-).$$

One can check that this is continuous if f and σ are continuous. Rewording slightly, we have proved the following theorem:

Theorem 2.1. *Suppose $\sigma \in C(X)$ and $f \in C(\Gamma_-^\theta)$. Then (2.2) has a unique solution $u_\theta \in C(X)$. Moreover u_θ has the explicit expression*

$$u_\theta(x) = \exp\left(-\int_{L_x} \sigma(t)\, dt\right) f(x^-),$$

where x_- is the point in Γ_-^θ obtained by travelling in the $-\theta$ direction from x, and L_x is the line between x and x^-.

2.3. **X-Ray Transform.** The solution to the forwards problem tells us something very important about the albedo map: if we pick a boundary condition $f \in \Gamma^\theta_-$ and consider a point $x^+ \in \Gamma^{\theta_+}$, then

$$\mathcal{A}(f)(x^+) = u_\theta(x^+) = \exp\left(-\int_L \sigma(t)\,dt\right) f(x^-),$$

where x_- is the corresponding point in Γ^θ_- and L is the line from x^- to x^+! If $f(x^-)$ is nonzero (and we get to pick it, so why not) we can divide by $f(x^-)$ and take logs to get

$$\int_L \sigma(t)dt.$$

Note that we can do this for any line L!

If we define $\mathcal{L}$ to be the set of all lines in $\mathbb{R}^n$ that pass through X, we can define the X-ray transform of σ to be the function on $\mathcal{L}$ given by

$$\mathcal{X}[\sigma](L) = \int_L \sigma(t)\,dt.$$

By the discussion above we've reduced our original inverse problem to this one: we need to show that given the X-ray transform of σ, we can recover σ.

The X-ray transform is slightly tricky to deal with (in dimensions 3 and higher) so as a warm-up exercise we will try something simpler and historically older: the Radon transform.

2.4. **The Radon Transform.** Let $\mathcal{H}$ be the set of all $n-1$ dimensional (hyper)planes in $\mathbb{R}^n$. Note that each hyperplane $H \in \mathcal{H}$ has an expression of the form

$$H = \{x \in \mathbb{R}^n | x \cdot \alpha = s\}$$

for some $\alpha \in S^{n-1}$ and $s \in \mathbb{R}$. This expression is not unique: the hyperplane corresponding to (α, s) also corresponds to $(-\alpha, -s)$. But certainly every hyperplane in $\mathcal{H}$ has a representation of this form. We could parametrize $\mathcal{H}$ by $S^{n-1} \times \mathbb{R}$ quotiented by the equivalence relation $(\alpha, s) = (-\alpha, -s)$. But then $S^{n-1} \times \mathbb{R}$ is an orientable double cover of $\mathcal{H}$, and as we'll see we want to take integrals, it makes sense to work on this space instead.

Definition 2.1. *The Radon transform is a map $\mathcal{R} : C_0(\mathbb{R}^n) \to C_0(S^{n-1} \times \mathbb{R})$ defined by*

$$\mathcal{R}[u](\alpha, s) = \int_{\{x\cdot\alpha = s\}} u(y)\,dH_y$$

The fact that $\mathcal{R}[u]$ is continuous if u is continuous is left as an exercise (hint: don't do this exercise).

Note that it is easy enough to specify a compact set X and define the Radon transform as a map defined on $C(X)$ instead: the treatment of the two maps is basically identical.

As a final observation, note that if we know the X-ray transform of σ then we know the Radon transform of σ, just by writing each plane as a disjoint union of parallel lines. In fact in two dimensions, the Radon transform and the X-ray transform are identical! And in principle one could do a reconstruction in higher dimensions by considering each two dimensional slice only. So in principle it suffices to understand the Radon transform. (This is true if we are only concerned about uniqueness. If we are concerned about obtaining the best possible reconstruction, we would want to consider the X-ray transform on its own.)