1. January 17

1.1. **Adjoint transform.** As we said earlier, two of the best technical tools we have in this class are the Fourier transform and integration by parts. Let's try to understand how the Radon transform interacts with these two tools.

We'll start with "integration by parts". Strictly speaking, you can't integrate by parts with the Radon transform since it isn't a derivative operator. But integration by parts is really a state of mind: if you view the integral $\int uv$ as being an inner product of u and v, then integration by parts is really about passing from an operator to its adjoint.

Let's try to see if we can make any sense of this with the Radon transform. We are looking for an operator $\mathcal{R}^*$ such that

$$\int \mathcal{R}[u]v\, d\alpha\, ds = \int u\mathcal{R}^*[v]\, dx.$$

What sets are we integrating over? We're applying the Radon transform to u, so u must be a function on $\mathbb{R}^n$ and then $R[u]$ is a function on $S^{n-1} \times \mathbb{R}$. Then for these integrals to make sense, v must be a function on $S^{n-1} \times \mathbb{R}$, and $R^*[v]$ is a function on $\mathbb{R}^n$. Therefore we're looking for an operator R^* that maps functions on $S^{n-1} \times \mathbb{R}$ to functions on $\mathbb{R}^n$, with the property that

$$\int_{S^{n-1}\times\mathbb{R}} \mathcal{R}[u]v\, d\alpha\, ds = \int_{\mathbb{R}^n} u\mathcal{R}^*[v]\, dx. \tag{1.1}$$

Let's write out the left side in full:

$$\int_{S^{n-1}\times\mathbb{R}} \mathcal{R}[u]v\, d\alpha\, ds = \int_{\mathbb{R}}\int_{S^{n-1}}\int_{\{x\cdot\alpha=s\}} \sigma(y)\, v(\alpha, s) dH_y\, d\alpha\, ds.$$

Since y is forced to be in the set $\{x \cdot \alpha = s\}$, we can write $s = y \cdot \alpha$ to get

$$\int_{S^{n-1}\times\mathbb{R}} \mathcal{R}[u]v\, d\alpha\, ds = \int_{\mathbb{R}}\int_{S^{n-1}}\int_{\{x\cdot\alpha=s\}} \sigma(y)\, v(\alpha, y\cdot\alpha) dH_y\, d\alpha\, ds.$$

By Fubini, we can rewrite as

$$\int_{S^{n-1}\times\mathbb{R}} \mathcal{R}[u]v\, d\alpha\, ds = \int_{S^{n-1}}\int_{\mathbb{R}}\int_{\{x\cdot\alpha=s\}} \sigma(y)\, v(\alpha, y\cdot\alpha)\, dH_y\, ds\, d\alpha.$$

Now there's an important observation to make: for a fixed value of α, the $dH_y\, ds$ integral is really an integral over all of $\mathbb{R}^n$. To see this, note that the dH_y integral is an integral over a plane perpendicular to α, and the ds integral indexes all the planes perpendicular to α, so combining these integrals gives the integral over $\mathbb{R}^n$. Therefore

$$\int_{S^{n-1}\times\mathbb{R}} \mathcal{R}[u]v\, d\alpha\, ds = \int_{S^{n-1}}\int_{\mathbb{R}^n} \sigma(y)\, v(\alpha, y\cdot\alpha)\, dy\, d\alpha.$$

Using Fubini again, we have

$$\begin{aligned}\int_{S^{n-1}\times\mathbb{R}} \mathcal{R}[u]v\,d\alpha\,ds &= \int_{\mathbb{R}^n}\int_{S^{n-1}} \sigma(y)\,v(\alpha, y\cdot\alpha)\,d\alpha\,dy \\ &= \int_{\mathbb{R}^n} \sigma(y)\left(\int_{S^{n-1}} v(\alpha, y\cdot\alpha)\,d\alpha\right)\,dy\end{aligned}$$

Comparing this to (1.1), we can see that the operator we're looking for is:

Definition 1.1. *The adjoint Radon transform (or backprojection) is the operator* R^* : $C(S^{n-1}\times\mathbb{R}) \to C(\mathbb{R}^n)$ *defined by*

$$R^*[v](x) = \int_{S^{n-1}} v(\alpha, x\cdot\alpha)\,d\alpha.$$

One note on the domain of definition. Since $\mathcal{R}^*$ is the integral over a compact set, it's not really necessary to limit the domain of $\mathcal{R}^*$ to compactly supported functions.

What is the meaning of this? If you fix an $x_0 \in \mathbb{R}^n$ and consider the set of planes of the form $\{x\cdot\alpha = s\}$ that pass through x_0, you should be able to convince yourself that any α is allowed, so long as $s = x_0\cdot\alpha$. In other words, $R^*[v](x)$ is the integral of v over all lines that pass through x! This provides a neat symmetry between R and its adjoint.

Note that if $u \in C_0(\mathbb{R}^n)$ is a nonnegative function supported in some ball B, then $R^*[R[u]](x)$ is large for $x \in B$ (since all lines through x have some nontrivial intersection with B) and small for x far from B (since very few lines through x will intersect B). This suggests that a) R^* can be used as a sort of second-rate "nearly inverse" operator and b) the actual inverse might involve R^*. Both of these are true and give some reason for the radiologists' name (backprojection) for R^*.

1.2. **Fourier Slice Theorem.** Now let's turn to the Fourier transform. It's not really clear at first glance how to define the Fourier transform of $\mathcal{R}[u](\alpha, s)$ since this is a function defined on $S^{n-1}\times\mathbb{R} \neq \mathbb{R}^n$. But what we can do is look at the Fourier transform in the s variable only:

$$\widehat{\mathcal{R}[u]}(\alpha,\sigma) = \int_{\mathbb{R}} \mathcal{R}[u](\alpha, s)e^{-is\sigma}\,ds$$

Note that the integral on the right is well defined: since u is compactly supported, for any fixed α, the function $\mathcal{R}[u](\alpha, s)$, viewed as a function of s, is in $C_0(\mathbb{R})$.

Writing out $\mathcal{R}$ explicitly, we get

$$\begin{aligned}\widehat{\mathcal{R}[u]}(\alpha,\sigma) &= \int_{\mathbb{R}}\left(\int_{\{x\cdot\alpha=s\}} u(y)\,dH_y\right)e^{-is\sigma}\,ds \\ &= \int_{\mathbb{R}}\int_{\{x\cdot\alpha=s\}} u(y)e^{-is\sigma}\,dH_y\,ds \\ &= \int_{\mathbb{R}}\int_{\{x\cdot\alpha=s\}} u(y)e^{-iy\cdot\alpha\sigma}\,dH_y\,ds\end{aligned}$$

We'll take advantage of the same important observation we made above: the $dH_y\, ds$ integral is actually the integral over $\mathbb{R}^n$. This tells us that

$$\begin{aligned}\widehat{\mathcal{R}[u]}(\alpha, \sigma) &= \int_{\mathbb{R}^n} u(y) e^{-iy\cdot\alpha\sigma} dy \\ &= \hat{u}(\sigma\alpha).\end{aligned}$$

This is the Fourier Slice Theorem:

Theorem 1.1. *Suppose $u \in C_0(\mathbb{R}^n)$. Then for any fixed α, the function $\mathcal{R}[u](\alpha, s)$, viewed as a function of s, is in $C_0(\mathbb{R})$, and*

$$\widehat{\mathcal{R}[u]}(\alpha, \sigma) = \hat{u}(\sigma\alpha).$$

Great! That works out better than we had any right to expect. What's going on here? One way to understand this is that if you take the Fourier transform of u at $\sigma\alpha$, you're multiplying u by a wave that's constant in directions perpendicular to α, and then integrating the whole thing up. In other words, you're just integrating along planes perpendicular to α (i.e., taking a Radon transform), and then taking the Fourier transform of what's left. That's the Fourier slice theorem in a nutshell.

Here's a neat consequence:

Corollary 1.2. *Suppose $u \in C_0^2(\mathbb{R}^n)$. Then*

$$\partial_s^2 \mathcal{R}[u](\alpha, s) = \mathcal{R}[\triangle u](\alpha, s).$$

Proof. Take the Fourier transform of $\mathcal{R}[\triangle u]$. By the Fourier slice theorem,

$$\begin{aligned}\widehat{\mathcal{R}[\triangle u]}(\alpha, \sigma) &= \widehat{\triangle u}(\sigma\alpha) \\ &= -|\sigma\alpha|^2 \hat{u}(\sigma\alpha) \\ &= -\sigma^2 \hat{u}(\sigma\alpha).\end{aligned}$$

Using the Fourier slice theorem again,

$$\begin{aligned}-\sigma^2 \hat{u}(\sigma\alpha) &= -\sigma^2 \widehat{\mathcal{R}[u]}(\alpha, \sigma) \\ &= \widehat{\partial_s^2 \mathcal{R}[u]}(\alpha, \sigma).\end{aligned}$$

Therefore

$$\widehat{\mathcal{R}[\triangle u]}(\alpha, \sigma) = \widehat{\partial_s^2 \mathcal{R}[u]}(\alpha, \sigma)$$

and the result now follows. □

So the Radon transform can turn the (partial differential) operator $\triangle$ into the (ordinary differential) operator ∂_s^2. This is neat but not usually practical. One common example of a place where it is practical is in analysis of the wave equation $(\partial_t - \triangle)u(x, t) = f$. As veterans of MA 533 know, this is easy to solve in one spatial dimension (d'Alembert formula!) but hard in higher dimensions. With the Radon transform, one can reduce higher dimensional versions to the one dimensional version and solve using d'Alembert's formula. Inverting the Radon transform gives the full solution. Of course to do this one has to invert the Radon transform.

2. January 19

One more comment on Corollary 1.2 before we move on: it should be clear that if $u \in C_0^{2k}(\Omega)$ then

$$\partial_s^{2k}\mathcal{R}[u](\alpha, s) = \mathcal{R}[\triangle^k u](\alpha, s).$$

In particular, we have the following corollary. (Maybe it's a synecdoche?.)

Corollary 2.1. *Suppose $u \in C_0^{2k}(\mathbb{R}^n)$. Then for any fixed α, the function $\mathcal{R}[u](\alpha, s)$, as a function of s, is in $C_0^{2k}(\mathbb{R})$. In particular if $u \in C_0^{\infty}(\mathbb{R}^n)$ then as a function of s, $\mathcal{R}[u](\alpha, s) \in C_0^{2k}(\mathbb{R})$.*

Actually, there's nothing special about $2k$ here: Corollary 1.2 is really just a special case of

$$|\sigma|^k \widehat{\mathcal{R}[u]}(\alpha, \sigma) = |\sigma|^k \hat{u}(\sigma\alpha).$$

with $k = 2$. If we define the operator $\sqrt{-\triangle}$ (or $(-\triangle)^{\frac{1}{2}}$) by

$$\widehat{\sqrt{-\triangle}u}(\xi) = |\xi|\hat{u}(\xi)$$

(note: what a great definition! we're using the Fourier transform to define *functions* of $\triangle$) then a more general version of Corollary 1.2 says that

$$(-\triangle_s)^{k/2}\mathcal{R}[u](\alpha, s) = \mathcal{R}[(-\triangle_x)^{k/2}u](\alpha, s).$$

As in Corollary 1.2, that's the one-dimensional Laplacian $\triangle_s = \partial_s^2$ on the right and the n-dimensional Laplacian $\triangle_x$ on the right. When k is even, this is a statement about honest differential operators. When k is not even, these operators are less well behaved. They're non-local, for a start, but more on this later.

2.1. Uniqueness.

Ok, with all these tools in place, let's solve some inverse problems. First up is the question of uniqueness. If $\mathcal{R}[u_1] = \mathcal{R}[u_2]$, does it follow that $u_1 = u_2$?

Theorem 2.2. *Yes.*

Proof. First note that the Radon transform is linear, so

$$\mathcal{R}[u_1] - \mathcal{R}[u_2] = \mathcal{R}[u_1 - u_2].$$

Therefore it suffices to show that if $\mathcal{R}[u] \equiv 0$ then $u = 0$. But if $\mathcal{R}[u] \equiv 0$ then the Fourier slice theorem implies that $\hat{u} \equiv 0$ so $u \equiv 0$.

□

That was easy. Next up: an inversion formula?

2.2. **Inversion.** Suppose $u \in C_0^\infty(\mathbb{R}^n)$. We want a formula for obtaining u from $R[u]$. The key is the Fourier slice theorem, which says that

$$\widehat{\mathcal{R}[u]}(\alpha,\sigma) = \hat{u}(\sigma\alpha).$$

Why is this key? Well we already know how to invert the Fourier transform:

$$u(x) = \int_{\mathbb{R}^n} \hat{u}(\xi)e^{ix\cdot\xi}d\xi.$$

(*mumble mumble 2π mumble mumble*) If we wrote this in spherical coordinates we'd get

$$u(x) = \int_0^\infty \int_{S^{n-1}} \hat{u}(\sigma\alpha)e^{ix\cdot\sigma\alpha}\sigma^{n-1}\,d\alpha\,d\sigma. \tag{2.1}$$

Now the integrand looks familiar: it contains the expression on the right side of the Fourier slice theorem! In retrospect the Fourier slice theorem is basically crying out to be integrated in spherical coordinates: we have $x \in \mathbb{R}^n$ on the right side expressed as an angle α times a distance σ. So let's do it: we have

$$u(x) = \int_0^\infty \int_{S^{n-1}} \widehat{\mathcal{R}[u]}(\alpha,\sigma)e^{ix\cdot\sigma\alpha}\sigma^{n-1}\,d\alpha\,d\sigma.$$

The most straightforward thing to do now would be to expand out that Fourier transform on the right side:

$$u(x) = \int_0^\infty \int_{S^{n-1}} \int_{\mathbb{R}} \mathcal{R}[u](\alpha,s)e^{i\sigma(x\cdot\alpha-s)}\sigma^{n-1}\,ds\,d\alpha\,d\sigma. \tag{2.2}$$

It's an inversion formula! It's a little funny looking though – it's not really built out of meaningful operators. In particular it's not really obvious how we'd analyze the stability of this thing. So let's go back to (2.1) and see if we can do better.

The worst thing about (2.2) is that to invert the Fourier transform we need to introduce a new integral. The existing $d\sigma$ integral is almost enough to invert the Fourier transform: if we multiply a function by $e^{ix\cdot\sigma\alpha}$ and integrate in σ, then we're really taking the inverse Fourier transform and evaluating at $x\cdot\alpha$. The catch is that the $d\sigma$ integral is only over half of the real line.

How do we fix this? Note that we could go back to (2.1) and equally write

$$u(x) = \int_{-\infty}^0 \int_{S^{n-1}} \hat{u}(\sigma\alpha)e^{ix\cdot\sigma\alpha}(-\sigma)^{n-1}\,d\alpha\,d\sigma$$

instead. Combining this with (2.1) we have

$$u(x) = \frac{1}{2}\int_{-\infty}^\infty \int_{S^{n-1}} \hat{u}(\sigma\alpha)e^{ix\cdot\sigma\alpha}|\sigma|^{n-1}\,d\alpha\,d\sigma.$$

Now let's sub in the Fourier slice theorem from here:

$$u(x) = \frac{1}{2}\int_{-\infty}^\infty \int_{S^{n-1}} \widehat{\mathcal{R}[u]}(\alpha,\sigma)e^{ix\cdot\sigma\alpha}|\sigma|^{n-1}\,d\alpha\,d\sigma.$$

Thanks to the discussion following Corollary 1.2, we have a ready-made interpretation of $|\sigma|^{n-1}\widehat{\mathcal{R}[u]}(\alpha,\sigma)$: it's just

$$\widehat{(-\triangle_s)^{(n-1)/2}}\mathcal{R}[u](\alpha,\sigma).$$

(Now you see why we wanted $u \in C_0^\infty(\mathbb{R}^n)$)! Therefore

$$u(x) = \frac{1}{2}\int_{-\infty}^{\infty}\int_{S^{n-1}} \widehat{(-\triangle_s)^{(n-1)/2}}\mathcal{R}[u](\alpha,\sigma)e^{ix\cdot\sigma\alpha}\,d\alpha\,d\sigma.$$

If we rewrite this as

$$u(x) = \frac{1}{2}\int_{S^{n-1}}\int_{\mathbb{R}} \widehat{(-\triangle_s)^{(n-1)/2}}\mathcal{R}[u](\alpha,\sigma)e^{i\sigma(x\cdot\alpha)}\,d\sigma\,d\alpha,$$

we can see that the $d\sigma$ integral is an inverse Fourier transform, evaluated at $x\cdot\alpha$: exactly what we wanted.

$$u(x) = \frac{1}{2}\int_{S^{n-1}} (-\triangle_s)^{(n-1)/2}\mathcal{R}[u](\alpha, x\cdot\alpha)\,d\alpha.$$

Do you recognize this integral? It's just the backprojection!

$$u(x) = \frac{1}{2}\mathcal{R}^*(-\triangle_s)^{(n-1)/2}\mathcal{R}[u](x).$$

Beautiful!