1. January 22

1.1. **Even and Odd.** In the previous class we proved the following theorem:

Theorem 1.1. *Suppose $u \in C_0^\infty(\mathbb{R}^n)$. Then*

$$u(x) = \frac{1}{2}\mathcal{R}^*(-\triangle)^{(n-1)/2}\mathcal{R}[u](x). \tag{1.1}$$

Note that this theorem acts very differently in even and in odd dimensions. In odd dimensions, $(-\triangle)^{(n-1)/2}$ is a genuine differential operator: it's just

$$(-\triangle)^{(n-1)/2} = -\partial_s^{n-1}.$$

Therefore we have

Theorem 1.2. *Suppose $u \in C_0^\infty(\mathbb{R}^n)$, where n is odd. Then*

$$u(x) = -\frac{1}{2}\mathcal{R}^*[\partial_s^{n-1}\mathcal{R}[u]](x).$$

On the other hand, if n is even, $(-\triangle)^{(n-1)/2}$ is not a differential operator. This is a real problem, since it's not necessarily clear that $(-\triangle)^{(n-1)/2}\mathcal{R}u$ is continuous, and so it's not really clear that we're allowed to apply the backprojection to it. Let's investigate: we need to break down the operator a little. For a function $f : \mathbb{R} \to \mathbb{R}$, we can write

$$\begin{aligned} \widehat{(-\triangle)^{(n-1)/2}} f(\sigma) &= |\sigma|^{n-1}\hat{f}(\sigma) \\ &= h(\sigma)(i\sigma)^{n-1}\hat{f}(\sigma) \end{aligned}$$

where

$$h(\sigma) = \begin{cases} i & \text{if } \sigma \geq 0 \\ -i & \text{if } \sigma < 0 \end{cases}$$

Therefore

$$\widehat{(-\triangle)^{(n-1)/2}} f(\sigma) = -h(\sigma)\widehat{\partial_s^{n-1} f}(\sigma).$$

In other words

$$(-\triangle)^{(n-1)/2} f(s) = -H\partial_s^{n-1} f(s),$$

where H is the Hilbert transform

$$\widehat{Hf}(\sigma) = h(\sigma)\hat{f}(\sigma).$$

It turns out that H maps smooth compactly supported functions to smooth functions (this is a consequence of the general fact that H maps H^k to H^k for any k, plus Sobolev embedding). This is great news: it tells us that Theorem 1.1 works even in even dimensions.

We could write the even-dimensional theorem as follows:

Theorem 1.3. *Suppose $u \in C_0^\infty(\mathbb{R}^n)$, where n is even. Then*

$$u(x) = -\frac{1}{2}\mathcal{R}^*[H\partial_s^{n-1}\mathcal{R}[u]](x).$$

This H makes a huge difference, though. First of all, H isn't a local operator. That means to apply this reconstruction formula at $x \in \mathbb{R}^n$, you need to know $\mathcal{R}[u](\alpha, s)$ for all α and s!. This is much worse than the odd-dimensional case. Note that as a happy accident, we live in odd-dimensional space! This means we can choose to ignore the problems with even-dimensional reconstruction if we want.

(The Hilbert transform has a long, noble, and independent history that goes far beyond its bit part in these notes. Often the Hilbert transform is written as

$$Hf(s) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(t)}{s-t}\, dt, \tag{1.2}$$

which turns out to be the same thing. The Hilbert transform maps L^p to L^p for $1 < p < \infty$. For a long time the only known proofs of this fact involved complex analysis, and they didn't generalize to other kinds of singular integrals. The quest for a real analysis proof of this fact that generalized to a large class of singular integrals was a big problem in the first half of the twentieth century; it was solved in large part thanks to Alberto Calderón, who also wrote about the inverse problem that bears his name. Among the vast majority of analysts, Calderón's contribution to the theory of singular integrals is much much better known than his contribution to inverse problems. Ok. Back to Radon transforms now.)

One serious issue with Theorem 1.1 and its parity-conscious relatives is that it only applies to smooth functions. In general, real world functions are not smooth! We can do a little better by tracking exactly how smooth we need our functions to be (really, in odd dimensions, we could get away with $n-1$ derivatives). But real world functions are not $n-1$ times differentiable either.

To do better, we need to expand the domain of $\mathcal{R}$ and its adjoint. How would we do this? One way would be to exploit the duality of $\mathcal{R}$ and its adjoint. For example, for $u \in L^2(\mathbb{R}^n)$, we could define $\mathcal{R}[u] = w$ if

$$\int_{\mathbb{R}^n} u\mathcal{R}^*[v]\, dx = \int_{S^{n-1}\times\mathbb{R}} wv\, dx \tag{1.3}$$

for all $v \in C_0(S^{n-1} \times \mathbb{R})$. This is not a bad idea! But it leaves some room for doubt: how do we know that any such w exists? Actually it's possible to show it exists by Hahn-Banach and the Riesz representation theorem, (if you don't follow this part, don't worry: just mentally replace this by "by appeal to the spirits of functional analysis") but this doesn't answer many questions: it shows that it exists but not what it is or how to find it. Let's leave this question for now and take a moment to think about stability.

1.2. **Stability.** If you believe in Theorem 1.1 then a stability analysis practically writes itself. Let's see how this works. First of all, the presence of the adjoint operator is really hinting that we should look at the inner product of (1.1) with something. So let's do that:

for $u, v \in C_0^\infty$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} uv\,dx &= \frac{1}{2}\int_{\mathbb{R}^n} \mathcal{R}^*(-\triangle)^{(n-1)/2}\mathcal{R}[u]v\,dx \\ &= \frac{1}{2}\int_{S^{n-1}\times\mathbb{R}} (-\triangle)^{(n-1)/2}\mathcal{R}[u]\mathcal{R}[v]\,d\alpha\,ds.\end{aligned}$$

Can you see the stability estimate from here? If we let $v = u$, we get

$$\|u\|_{L^2(\mathbb{R}^n)} = \frac{1}{2}\int_{\mathbb{R}^n} (-\triangle)^{(n-1)/2}\mathcal{R}[u]\mathcal{R}[u]\,d\alpha\,ds.$$

The operator $(-\triangle)^{(n-1)/2}$ is the square of the self adjoint operator $(-\triangle)^{(n-1)/4}$, so

$$\|u\|^2_{L^2(\mathbb{R}^n)} = \frac{1}{2}\|(-\triangle)^{(n-1)/4}\mathcal{R}[u]\|^2_{L^2(\mathbb{R}^n)}.$$

(Aside: if it's not clear what I mean here, use the Plancherel theorem to replace the inner product on the right with the inner product of the Fourier transforms:

$$\begin{aligned}\|u\|^2_{L^2(\mathbb{R}^n)} &= \frac{1}{2}\int_{S^{n-1}\times\mathbb{R}} |\xi|^{n-1}\widehat{\mathcal{R}[u]}\widehat{\mathcal{R}[u]}\,d\alpha\,ds \\ &= \frac{1}{2}\int_{S^{n-1}\times\mathbb{R}} \left(|\xi|^{(n-1)/2}\widehat{\mathcal{R}[u]}\right)^2\,d\alpha\,ds \\ &= \frac{1}{2}\|(-\triangle)^{(n-1)/4}\mathcal{R}[u]\|^2_{L^2(S^{n-1}\times\mathbb{R})}\end{aligned}$$

Another approach is to throw away the even dimensions, in which case n is odd and $(-\triangle)^{(n-1)/2}$ is the differential operator $-\partial_s^{n-1} = -(\partial_s^{(n-1)/2})^2$. Then we can integrate by parts in the honest Calc II way, and everything makes sense. End of Aside.)

Let's record this result as a theorem.

Theorem 1.4. *There exists a constant C such that*

$$\|u\|_{L^2(\mathbb{R}^n)} = C\|(-\triangle)^{(n-1)/4}\mathcal{R}[u]\|_{L^2(S^{n-1}\times\mathbb{R})} \tag{1.4}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$.

This has two important consequences. The first is the result we came here for.

Corollary 1.5. *For all $u_1, u_2 \in C_0^\infty(\mathbb{R}^n)$,*

$$\|u_1 - u_2\|_{L^2(\mathbb{R}^n)} \lesssim \|\mathcal{R}[u_1] - \mathcal{R}[u_2]\|_{\dot{H}_s^{(n-1)/2)}(S^{n-1}\times\mathbb{R})}. \tag{1.5}$$

In particular

$$\|u_1 - u_2\|_{L^2(\mathbb{R}^n)} \lesssim \|\mathcal{R}[u_1] - \mathcal{R}[u_2]\|_{H_s^{(n-1)/2)}(S^{n-1}\times\mathbb{R})}. \tag{1.6}$$

Here the norms $H_s^{(n-1)/2)}(S^{n-1}\times\mathbb{R})$ and $\dot{H}_s^{(n-1)/2)}(S^{n-1}\times\mathbb{R})$ are Sobolev norms in the s variable:

$$\|u\|_{H_s^{(n-1)/2)}(S^{n-1}\times\mathbb{R})} = \|(1+|\sigma|)^{(n-1)/2}\hat{u}\|_{L^2(S^{n-1}\times\mathbb{R})},$$

and the homogeneous variant is described equivalently. (Remember that $\hat{\cdot}$ in this case is a Fourier transform only in the s variable!)

To reinterpret Corollary 1.5, if $\mathcal{R}[u_1]$ and $\mathcal{R}[u_2]$ are close in the $H_s^{(n-1)/2)}(S^{n-1} \times \mathbb{R})$ norm, then u_1 and u_2 are close in the L^2 norm. The proof of Corollary 1.5 follows from Theorem 1.4 and the linearity of the Radon transform.

Some notes on this stability estimate. First, (1.5) is really a slightly better estimate than (1.6). In practice, though, we're usually only concerned with functions supported on a fixed compact subset of $\mathbb{R}^n$, in which case Poincaré's inequality makes this observation more or less irrelevant.

Secondly, as in the discussion of Theorem 1.1, Corollary 1.5 has separate interpretations in the even and odd dimensional cases. In the odd dimensional cases, following the argument in the aside leads to the very nice special case of (1.4):

$$\|u\|_{L^2(\mathbb{R}^n)} = \frac{1}{2}\|\partial_s^{(n-1)/2}\mathcal{R}[u]\|_{L^2(\mathbb{R}^n)}.$$

In even dimensional cases nothing so nice happens.

Thirdly, note that the stability estimate involves *derivatives*. This is *bad*. If our observations contain some noise, and the noise is not differentiable, then in principle our reconstruction could be really bad. However, some philosophical reflection should convince you that to a certain extent this is inevitable; taking the Radon transform involves integrating the function up, so it's not surprising that reconstruction depends on the derivatives of the Radon transform. In practice this turns out to be not so fatal.

Fourthly, note that this is an L^2 stability estimate. L^2 isn't necessarily the best space for stability estimates in practice: it's possible to have huge artifacts in the reconstruction. We proved an L^2 estimate because the light is better there, as the old joke goes. There is relatively modern work (21st century at least) that looks at stability in other norms.

The fifth note is the most important. Why do we believe that Corollary 1.5 is meaningful? If $\mathcal{R}u$ is not in the Sobolev space $H_s^{(n-1)/2}$ then (1.6) is really telling us that $\|u\|_{L^2(\mathbb{R}^n)} \leq \infty$, which is not really useful information. This is starting to look very bad. But (1.4) is here to save us: it proves a second theorem as well.

2. January 24

Corollary 2.1. *Suppose $u \in C_0^\infty(\mathbb{R}^n)$. Then $\mathcal{R}[u] \in \dot{H}_s^{(n-1)/2)}(S^{n-1} \times \mathbb{R})$, and*

$$\|\mathcal{R}[u]\|_{\dot{H}_s^{(n-1)/2)}(S^{n-1}\times\mathbb{R})} \lesssim \|u\|_{L^2(\mathbb{R}^n)}.$$

Moreover, if $u \in C_0^\infty(X)$ for some compact set X, then there exists a constant C_X depending only on X so that

$$\|\mathcal{R}[u]\|_{H_s^{(n-1)/2)}(S^{n-1}\times\mathbb{R})} \leq C_X\|u\|_{L^2(\mathbb{R}^n)}.$$

The proof is straight out of Theorem (1.4), with the last part following from a Poincaré inequality.

Note that it tells us immediately that Corollary (1.5) has meaning: the right side of the estimates makes sense.

But it also has an important consequence: it tells us that we can extend the definition of $\mathcal{R}$ by a *density argument*.

Suppose $u \in L^2(\mathbb{R}^n)$. Let $\{u_i\}$ be a sequence of $C_0^\infty(\mathbb{R}^n)$ functions that converges to u in the L^2 norm. The Radon transform of each $\{u_i\}$ is defined, and Corollary 2.1 tells us that

$$\|\mathcal{R}[u_i - u_j]\|_{\dot{H}_s^{(n-1)/2}(S^{n-1}\times\mathbb{R})} \lesssim \|u_i - u_j\|_{L^2(\mathbb{R}^n)}.$$

Now $\{u_i\}$ is Cauchy in the L^2 norm, so this tells us that $\mathcal{R}[u_i]$ is Cauchy in the $\dot{H}_s^{(n-1)/2}(S^{n-1}\times\mathbb{R})$ norm. Therefore it converges to some function $w \in \dot{H}_s^{(n-1)/2}(S^{n-1}\times\mathbb{R})$.

We'll define $w = \mathcal{R}[u]$. We really ought to check that this is independent of the Cauchy sequence we chose. But this is easy enough: if $\{u_i\}$ and $\{v_i\}$ both converge to u in L^2 norm, then $\|u_i - v_i\|_{L^2(\mathbb{R}^n)} \to 0$. Therefore Corollary 2.1 tells us that

$$\|\mathcal{R}[u_i] - \mathcal{R}[v_i]\|_{\dot{H}_s^{(n-1)/2}(S^{n-1}\times\mathbb{R})} \to 0.$$

Therefore $\mathcal{R}[u_i]$ and $\mathcal{R}[v_i]$ converge to the same function.

This justifies the following definition

Definition 2.1. *Suppose $u \in L^2(\mathbb{R}^n)$. Let $\{u_i\}$ be a sequence of $C_0^\infty(\mathbb{R}^n)$ functions that converges to u in the L^2 norm. We define $\mathcal{R}[u] \in \dot{H}_s^{(n-1)/2}(S^{n-1}\times\mathbb{R})$ to be*

$$\lim_{i\to\infty} \mathcal{R}[u_i].$$

This extends $\mathcal{R}$ to an operator

$$\mathcal{R} : L^2(\mathbb{R}^n) \to \dot{H}_s^{(n-1)/2}(S^{n-1}\times\mathbb{R}).$$

Note that the second part of Corollary (2.1) also lets us extend

$$\mathcal{R} : L^2(X) \to H_s^{(n-1)/2}(S^{n-1}\times\mathbb{R}),$$

for a compact set X.

Having extended $\mathcal{R}$, we could extend $\mathcal{R}^*$ in a similar way. But duality is more powerful here: we can define for $v \in L^2(S^{n-1}\times\mathbb{R})$, $\mathcal{R}^*[v] = w$ if

$$\int_{S^{n-1}\times\mathbb{R}} \mathcal{R}[u]v\,dx = \int_{\mathbb{R}^n} uw\,dx \tag{2.1}$$

for each $u \in L^2(\mathbb{R}^n)$. Why does w exist? Explicitly, we can tell that the map

$$u \mapsto \int_{S^{n-1}\times\mathbb{R}} \mathcal{R}[u]v\,dx$$

is a linear functional on $L^2(\mathbb{R}^n)$. The Riesz representation theorem tells us that this functional has a representation by some $w \in L^2(\mathbb{R}^n)$: that is, there exists $w \in \mathbb{R}^n$ such that

$$\int_{S^{n-1}\times\mathbb{R}} \mathcal{R}[u]v\,dx = \int_{\mathbb{R}^n} uw\,dx.$$

Fascinatingly, this argument still works for all v in the dual space to the space where $\mathcal{R}[u]$ lives. If we restrict ourselves to a compact set X, this has a very nice expression: the dual space to $H_s^{(n-1)/2)}(S^{n-1} \times \mathbb{R})$ is $H_s^{-(n-1)/2)}(S^{n-1} \times \mathbb{R})$. Therefore $\mathcal{R}^*$ gets extended to a map

$$\mathcal{R}^* : H_s^{-(n-1)/2)}(S^{n-1} \times \mathbb{R}) \to L^2(\mathbb{R}^n).$$

Extensions by density arguments and duality arguments are very common in analysis in general: veterans of MA 633 should recognize these from definitions of the derivative.

Using these two definitions of $\mathcal{R}$ on L^2, it is possible (though not necessarily easy!) to show that our previous theorems for invertibility and stability have much wider application. In their fullest form, you can show that

Theorem 2.2. *Suppose X is compact and $u \in L^2(X)$. Then*

$$u(x) = \frac{1}{2}\mathcal{R}^*(-\triangle)^{(n-1)/2}\mathcal{R}[u](x). \tag{2.2}$$

and

Corollary 2.3. *For all $u_1, u_2 \in L^2(X)$,*

$$\|u_1 - u_2\|_{L^2(X)} \lesssim \|\mathcal{R}[u_1] - \mathcal{R}[u_2]\|_{H_s^{(n-1)/2)}(S^{n-1}\times\mathbb{R})}. \tag{2.3}$$

I think we're done here.

2.1. **X-Ray Transform.** Everything we've done for the Radon transform turns out to have a parallel for the X-ray transform. With that in mind, let's not do the X-ray transform in too much detail. Instead we'll define it carefully, but just loosely describe the inversion process with no proofs.

How do we define it? First we need a parametrization of the set of lines $\mathcal{L}$ on $\mathbb{R}^n$. If we think about it carefully, we can define a line on $\mathbb{R}^n$ by specifying a direction $\alpha \in S^{n-1}$, and a point p on the plane $\mathbb{R}^{n-1}$ defined by $\{x \cdot \alpha = 0\}$. Then we can write the line through p in the direction of α as $\{x \in \mathbb{R}^n | x = p + t\alpha, t \in \mathbb{R}\}$.

As with the Radon transform, $S^{n-1} \times \mathbb{R}^{n-1}$ is a double cover of $\mathcal{L}$: the line specified by (α, p) is also the line specified by $(-\alpha, p)$. As with the Radon transform, let's not worry about this too much. (Actually, if you look at Uhlmann's notes, he treats the X-ray transform as a map into functions on $S^{n-1} \times \mathbb{R}$, which is much worse: it gives a $\mathbb{R}$'s worth of redundancy. This has advantages if you want to prove anything – it means you have to be less fearful of Fubini's theorem, for a start – but since I don't intend to prove anything let's forget this.)

Definition 2.2. *The X-ray transform is a map $\mathcal{X} : C_0(\mathbb{R}^n) \to C_0(S^{n-1} \times \mathbb{R}^{n-1})$ defined by*

$$\mathcal{X}[u](\alpha, p) = \int_{\{p+t\alpha\}} u(y)\, dt_y.$$

The X-ray transform has an adjoint, which corresponds to integration over all lines through x:

Definition 2.3. *Define $\mathcal{X}^* : C(S^{n-1} \times \mathbb{R}^{n-1}) \to C(\mathbb{R}^n)$ defined by*

$$\mathcal{X}^*[v](x) = \int_{S^{n-1}} v(\alpha, x^{\perp\alpha})\, d\alpha,$$

where $x^{\perp\alpha}$ *is the projection of* x *onto the plane* $\{x \cdot \alpha = 0\}$.

There's even a Fourier slice theorem: if we define $\widehat{\mathcal{X}[u]}(\alpha, \rho)$ to be the Fourier transform in the p variables, then we have

Theorem 2.4. *For* $u \in C_0(\mathbb{R}^n)$,

$$\widehat{\mathcal{X}[u]}(\alpha, \rho) = \hat{u}(\rho).$$

Finally, there's an inversion formula:

Theorem 2.5. *For* $u \in C_0^\infty(\mathbb{R}^n)$,

$$u(x) = \mathcal{X}^*(-\triangle)^{1/2}\mathcal{X}u(x).$$

The inversion formula has a related stability estimate:

Corollary 2.6. *For all* $u_1, u_2 \in C_0^\infty(\mathbb{R}^n)$,

$$\|u_1 - u_2\|_{L^2(\mathbb{R}^n)} \lesssim \|\mathcal{R}[u_1] - \mathcal{R}[u_2]\|_{\dot{H}_p^{1/2}(S^{n-1}\times\mathbb{R}^{n-1})}. \tag{2.4}$$

In particular

$$\|u_1 - u_2\|_{L^2(\mathbb{R}^n)} \lesssim \|\mathcal{R}[u_1] - \mathcal{R}[u_2]\|_{H_p^{1/2}(S^{n-1}\times\mathbb{R}^{n-1})}. \tag{2.5}$$

and as before, all of this extends naturally to $u \in L^2(\mathbb{R}^n)$. Details are left as an exercise to the reader.