1. January 26

Let's see what happens now if we add some *scattering*. The first step is to understand what a rudimentary scattering model looks like.

1.1. **Radiative Transport.**

Let's start by recalling the setup for the X-ray problem. We had a convex bounded open set $X \subset \mathbb{R}^n$ with smooth boundary; if you like you can pretend X is a ball.

For each direction $\theta \in S^{n-1}$ we had a function $u_\theta : X \to \mathbb{R}$ to represent the intensity of light in direction θ. Let's modify our notation a little bit: we'll write a single function $u : X \times S^{n-1} \to \mathbb{R}$, and let $u(x, \theta)$ represent the intensity of light at x in direction θ.

In the absence of scattering we had the governing equation

$$\theta \cdot \nabla_x u(x, \theta) = -\sigma(x) u(x, \theta).$$

The operator on the left is the directional derivative in the θ direction, so in other words, this tells us that we as we travel in the direction of θ, light in the θ direction drops off at a relative rate governed by the (nonnegative) absorption factor σ.

But now we have to add scattering. Scattering has two effects: it creates extra reduction of $u(x, \theta)$ in the θ direction, since some of the light that arrived at x by traveling in the θ direction now scatters out in some other direction. Therefore scattering increases the value of σ. But scattering also means that some of the light that arrives at x by traveling in some other direction now gets scattered into the θ direction. We model this by adding a

$$\int_{S^{n-1}} k(x, \theta, \theta') u(x, \theta') \, d\theta'$$

term onto the right side, where k is nonnegative. Therefore we get

$$\theta \cdot \nabla_x u(x, \theta) = -\sigma(x) u(x, \theta) + \int_{S^{n-1}} k(x, \theta, \theta') u(x, \theta') \, d\theta'. \tag{1.1}$$

This equation is sometimes called the radiative transport equation. (Sometimes it's called Boltzmann's equation, but Boltzmann's equation usually allows the particles to additionally travel at different speeds, so we have $u(x, v)$ where $v \in \mathbb{R}^n$. Of course photons all move at pretty much the same speed so our equation is a little nicer.)

To understand solutions to the RTE we need to be able to state boundary conditions. As in the non-scattering case, we need to define the incoming and outgoing boundaries: we define

$$\Gamma_\pm = \{(x, \theta) \in \partial X \times S^{n-1} | \pm \nu(x) \cdot \theta \geq 0\}$$

Roughly speaking, if $(x, \theta) \in \Gamma_-$ then θ points into X, and if $(x, \theta) \in \Gamma_+$ then θ points out. Therefore we can call Γ_- the incoming boundary of $X \times S^{n-1}$, and Γ_+ the outgoing boundary of $X \times S^{n-1}$.

Finally, we need to impose some conditions on the coefficients.

First, we'll specify that σ and k are nonnegative and continuous:

$$\sigma \in C(X), k \in C(X \times S^{n-1} \times S^{n-1}); \qquad \sigma, k \geq 0. \tag{1.2}$$

Continuity isn't really necessary but it'll make our lives slightly easier so let's run with it. Nonnegativity is physically appropriate, given what we intend σ and k to mean.

Second, we want to ensure that scattering doesn't have the effect of creating light. Therefore we need to specify that

$$\int_{S^{n-1}} \sigma(x)u(x,\theta)d\theta \geq \int_{S^{n-1}}\int_{S^{n-1}} k(x,\theta,\theta')u(x,\theta')\,d\theta' d\theta$$

for all reasonable nonnegative u. In other words

$$\sigma(x) \geq \left\| \int_{S^{n-1}} k(x,\theta,\theta')d\theta \right\|_{L^\infty(S^{n-1})}$$

for all $x \in X$. We'll ask for something slightly stronger: that there exists $c > 0$ such that

$$\inf_X \left(\sigma(x) - \left\| \int_{S^{n-1}} k(x,\theta,\theta')d\theta \right\|_{L^\infty(S^{n-1})} \right) > c. \tag{1.3}$$

Thirdly we will ask that k is isotropic:

$$k(x,\theta,\theta') = k(x,-\theta,-\theta') \tag{1.4}$$

This is a physically reasonable thing to ask (draw a diagram!) and it frequently makes our lives easier.

The following theorem guarantees the existence and uniqueness of solutions to (1.1).

Theorem 1.1. *Assume* (1.2), (1.3), *and* (1.4) *hold, and let* $f \in C(\Gamma_-)$. *Then the problem*

$$\begin{aligned} \theta\cdot\nabla_x u(x,\theta) &= -\sigma(x)u(x,\theta) + \int_{S^{n-1}} k(x,\theta,\theta')u(x,\theta')\,d\theta' \text{ on } X \\ u|_{\Gamma_-} &= f \end{aligned} \tag{1.5}$$

has a unique solution $u \in C(X \times S^{n-1})$.

We'll prove this later. First let's record the inverse problem. The basic inverse problem should be the following: we have a domain in which the light propagation is governed by (1.1). We can shine light into the domain (i.e., specify the incoming light $u|_{\Gamma_-}$, and measure the resulting outgoing light u_{Γ_+}. In other words we know the albedo map

$$\mathcal{A}_{\sigma,k} : C(\Gamma_-) \to C(\Gamma_+)$$

defined by

$$\mathcal{A}_{\sigma,k}(f) = u|_{\Gamma_+},$$

where u is the unique solution to (1.1) with boundary condition f. Theorem 1.1 guarantees that this map is well defined. We want to know if knowledge of $\mathcal{A}_{\sigma,k}$ determines σ and k. Note that reconstructing just one of σ and k suffices to provide imagery, so we could maybe settle for that if it's all we can do. A prominent physicist once told me that reconstructing σ is the real objective, and "reconstructing k is really just virtuoso stuff – something for mathematicians to do to show off."

2. January 29

2.1. Solving the RTE: Introduction. As in the X-ray problem, the key to solving the inverse problem is to understand the structure of solutions to the RTE.

We need to start by giving everything a name. Following Choulli-Stefanov, let's define

$$Au(x,\theta) = -\sigma(x)u(x,\theta) + \int_{S^{n-1}} k(x,\theta,\theta')u(x,\theta')\,d\theta',$$

and break A up into

$$A_1 u(x,\theta) = -\sigma(x)u(x,\theta)$$

and

$$A_2 u(x,\theta) = \int_{S^{n-1}} k(x,\theta,\theta')u(x,\theta')\,d\theta'.$$

Finally, let's define

$$Tu(x,\theta) = \theta \cdot \nabla_x u(x,\theta) + \sigma(x)u(x,\theta).$$

T is the critical operator here: it's the operator we know how to deal with. So let's write the RTE problem (1.5) as

$$\begin{aligned} (T - A_2)u &= 0 \text{ on } X \\ u|_{\Gamma_-} &= f \end{aligned} \tag{2.1}$$

In the grand tradition of analysis and PDE, let's begin by *ignoring the hard part* and seeing that we understand the problem in the so-called ballistic case, when $A_2 = 0$. We have two lemmas about the ballistic problem.

2.2. The Ballistic Equation. To make things slightly easier on ourselves, let's introduce two pieces of notation. First is the optical distance. The optical distance between two points x and y is the integral of σ along the line between x and y:

$$\tau(x,y) = \int_{L_(x,y)} \sigma(t)\,dt. \tag{2.2}$$

This quantity $\tau(x,y)$ is a measure of how hard the light has to try to get from the incoming boundary to (x,θ) in the absence of scattering. The integral is not oriented: $\tau(x,y)$ is the same as $\tau(y,x)$.

Secondly, let's define $\gamma_\pm(x,\theta)$ to be the intersection of ∂X with the ray from x in the $\pm\theta$ direction. (Draw a diagram!)

Ok. Here's the first lemma.

Lemma 2.1. *For $f \in C(\Gamma_-)$ define*

$$Jf(x,\theta) = e^{-\tau(x,\gamma_-(x,\theta))} f(\gamma_-(x,\theta),\theta). \tag{2.3}$$

Then $Jf \in C(X \times S^{n-1})$, and Jf is the unique solution to

$$\begin{aligned} TJf &= 0 \text{ on } X \\ Jf|_{\Gamma_-} &= f. \end{aligned}$$

Proof. You can check that Jf solves the given transport equation just by differentiating and checking the boundary condition. Deriving it is not much harder – you just need to solve the transport equation, which is secretly an ODE, in the right coordinates. In fact, we already proved this lemma back on January 10 – if you look at Theorem 2.1 in those notes and translate into our current notation, you get exactly the statement of this lemma. □

Here's the second lemma:

Lemma 2.2. *For $S \in C(X \times S^{n-1})$ define*

$$T^{-1}S(x,\theta) = \int_0^{|x-\gamma_-(x,\theta)|} e^{-\tau(x,x-t\theta)} S(x-t\theta,\theta)\, dt. \tag{2.4}$$

Then $T^{-1}S \in C(X \times S^{n-1})$, and $T^{-1}S$ is the unique solution to

$$TT^{-1}S = S \text{ on } X$$
$$T^{-1}S|_{\Gamma_-} = 0.$$

Proof. Again, checking that $T^{-1}S$ solves the given transport equation is just a matter of differentiating; deriving this solution is just a matter of picking the right coordinates and solving the ODE. □

2.3. **Solving the RTE.** .

Now that we know how to solve the ballistic case, we can let the other shoe drop: we'll treat the full problem (2.1) as a perturbation of the ballistic case. To do this successfully, we have to show that A_2 is somehow small with respect to the rest of the equation. It turns out (see the proof of Theorem 1.1, below, that the precise result we need is the following lemma.

Lemma 2.3. *There exists a constant c_1 with $0 < c_1 < 1$ such that*

$$\|T^{-1}A_2 u\| \le c_1 \|u\|_{C(X\times S^{n-1})}.$$

for all $u \in C(X \times S^{n-1})$.

Proof. First, we have

$$A_2 u(x,\theta) = \int_{S^{n-1}} k(x,\theta,\theta') u(x,\theta')\, d\theta',$$

so

$$|A_2 u(x,\theta)| \le \left| \int_{S^{n-1}} k(x,\theta,\theta') d\theta' \right| \sup_{\theta'} |u(x,\theta')|.$$

By (1.4) and the positivity of k,

$$|A_2 u(x,\theta)| \le \int_{S^{n-1}} k(x,\theta',-\theta) d\theta' \sup_{\theta'} |u(x,\theta')|.$$

Then (1.3) guarantees that

$$|A_2 u(x,\theta)| \le (\sigma(x) - c) \sup_{\theta'} |u(x,\theta')|$$

for some fixed positive c. Since σ is bounded on X, we could equally say

$$|A_2u(x,\theta)| \le c_1\sigma(x)\sup_{\theta'}|u(x,\theta')|$$

for some $0 < c_1 < 1$. Now

$$T^{-1}A_2u(x,\theta) = \int_0^{|x-\gamma_-(x,\theta)|} e^{-\tau(x,x-t\theta)}A_2u(x-t\theta,\theta)\,dt,$$

so

$$\begin{aligned} |T^{-1}A_2u(x,\theta)| &= \int_0^{|x-\gamma_-(x,\theta)|} e^{-\tau(x,x-t\theta)}c_1\sigma(x-t\theta)\sup_{\theta'}|u(x-t\theta,\theta')|\,dt \\ &\le c_1\int_0^{|x-\gamma_-(x,\theta)|} e^{-\tau(x,x-t\theta)}\sigma(x-t\theta)\,dt\|u\|_{C(X\times S^{n-1})} \end{aligned}$$

Remember what $\tau(x, x-t\theta)$ is: it's just the integral of σ along the straight line from x to $x - t\theta$. Therefore

$$\partial_t(-e^{-\tau(x,x-t\theta)}) = e^{-\tau(x,x-t\theta)}\sigma(x-t\theta).$$

This is super-convenient: it means we can integrate:

$$\begin{aligned} |T^{-1}A_2u(x,\theta)| &\le -e^{-\tau(x,x-t\theta)}\Big|_0^{|x-\gamma_-(x,\theta)|}c_1\|u\|_{C(X\times S^{n-1})} \\ &\le c_1\|u\|_{C(X\times S^{n-1})} \end{aligned}$$

and the result follows. □

Ok, now let's solve the full RTE.

Proof of Theorem 1.1. Using the notation from Section 2.1, we can write the RTE in Theorem 1.1 as (2.1):

$$\begin{gathered}(T-A_2)u = 0 \text{ on } X \\ u|_{\Gamma_-} = f\end{gathered}$$

Lemma 2.1 lets us zero out the boundary value: if u solves (2.1), and $\tilde{u} = u - Jf$, then

$$\begin{gathered}(T-A_2)\tilde{u} = A_2Jf \text{ on } X \\ \tilde{u}|_{\Gamma_-} = 0.\end{gathered} \tag{2.5}$$

Then Lemma 2.2 says we can apply T^{-1} to both sides of the equation to get

$$\begin{gathered}(I-T^{-1}A_2)\tilde{u} = T^{-1}A_2Jf \text{ on } X \\ \tilde{u}|_{\Gamma_-} = 0.\end{gathered} \tag{2.6}$$

(This last part is worth thinking about carefully: really what we're saying is that $\tilde{u}$ being the solution to (2.5) means that $\tilde{u}$ must be of the form $T^{-1}A_2(Jf+\tilde{u})$; this makes sense because Lemma 2.2 guarantees us that T^{-1} is the unique solution operator.)

Now we can solve (2.6) by taking a Neumann series: this is precisely what Lemma 2.3 guarantees us that we can do. Then

$$\tilde{u} = (I + T^{-1}A_2 + T^{-1}A_2T^{-1}A_2 + \ldots)T^{-1}A_2Jf$$

is the unique solution to (2.6). Substituting $u = \tilde{u} + Jf$, we get

$$u = (I + T^{-1}A_2 + T^{-1}A_2T^{-1}A_2 + \ldots)Jf. \tag{2.7}$$

□

3. January 31

Note that we've actually proved something stronger than Theorem 1.1: not only have we shown existence and uniqueness of solutions, we've actually written down a formula for the solution. And not only that: the formula gives us its own estimate: if c_1 is the constant from Lemma 2.3, then the formula for the sum of a geometric series tells us that

$$\|u\|_{C(X\times S^{n-1})} \leq \frac{1}{1-c_1}\|Jf\|_{C(X\times S^{n-1})}.$$

A short stare at the formula for J should convince you that

$$\|Jf\|_{C(X\times S^{n-1})} \leq \|f\|_{C(\Gamma_-)},$$

so

$$\|u\|_{C(X\times S^{n-1})} \leq \frac{1}{1-c_1}\|f\|_{C(\Gamma_-)}.$$

This estimate can take us a step further, actually, if we're willing to let it: if you note that the C norm is exactly the same as the L^∞ norm, then this tells us that for $f \in C(\Gamma_-)$, the function u defined by (2.7) satisfies the estimate

$$\|u\|_{L^\infty(X\times S^{n-1})} \leq \frac{1}{1-c_1}\|f\|_{L^\infty(\Gamma_-)}.$$

In the usual way, we can extend the map $f \mapsto u$ to a map on $L^\infty(\Gamma_-)$. You can check that this is still a meaningful solution of the RTE.

We can sum up this discussion by writing down an extra strong version of Theorem 1.1:

Theorem 3.1. *Assume* (1.2), (1.3), *and* (1.4) *hold, and let* $f \in L^\infty(\Gamma_-)$. *Then the problem*

$$(T - A_2)u = 0 \text{ on } X$$
$$u|_{\Gamma_-} = f$$

has a unique solution given by

$$u = (I + T^{-1}A_2 + T^{-1}A_2T^{-1}A_2 + \ldots)Jf. \tag{3.1}$$

Moreover

$$\|u\|_{L^\infty(X\times S^{n-1})} \lesssim \|f\|_{L^\infty(\Gamma_-)}. \tag{3.2}$$

The expansion (3.1) is sometimes called the *collision expansion* of u. Three more comments on this solution: First, the key reason the assumption (1.3) is needed is to ensure that $T^{-1}A_2$ is small enough to allow a Neumann series solution. But there are other ways to ensure this too. One notable way is to assume that the domain X is small. If we note that

$$|T^{-1}S(x,\theta)| \leq \int_0^{|x-\gamma_-(x,\theta)|} e^{-\tau(x,x-t\theta)}|S(x-t\theta,\theta)|\,dt \leq \|S\|_{L^\infty(X\times S^{n-1})}\mathrm{diam}X,$$

then it follows that for small enough X (relative to k), the operator $T^{-1}A_2$ is always small enough to allow a Neumann series solution. This makes some physical sense – even if a medium is highly scattering, the optical properties aren't bad as long as it's very thin relative to its scatteringness.

Secondly, there's a nice positivity result here: if f is nonnegative then so is u. This follows from the sign demands we made on k, and it's pleasingly intuitive: assuming you send light into X you won't ever get negative light inside X.

Finally, the collision expansion is physically meaningful, which shouldn't always be taken for granted. The first term (the ballistic term) corresponds, neatly enough, to the unscattered light: the solution in the absence of scattering.

What does the second term correspond to? Well, for any given (x,θ), the function $A_2Jf(x,\theta)$ means the following: take the light that arrived to x in the direction θ' without being scattered, and integrate up against k over all θ''s, and spit it out in the direction θ. Now the second term of the collision expansion is $T^{-1}A_2Jf$: it says that at (x,θ), we should take all the light along the ray from x in direction $-\theta$ that orginated in this way, and treat it as the source for an ballistic RTE. In other words, it says that $T^{-1}A_2Jf$ represents the integral of all the unscattered light that arrived onto the ray from x in direction $-\theta$, changed direction once and travelled to x without scattering again. (You really ought to draw a picture to represent this argument.) It represents *light that's scattered exactly once*!

Similarly the third term represents the light scattered exactly twice, and so on. This is great: it tells us that we can use our physical intuition about what the terms look like.

3.1. **Point Sources.** Let's test this intuition by trying to understand what happens when we have an (approximate) point source in $n = 3$ dimensions. (Digression: if you actually read the Choulli-Stefanov paper, you'll find they take Theorem 3.1 even further and show that solutions make sense even when the boundary source is a distribution. Then the notion of point sources can be done exactly: we can talk about using a δ function as the boundary condition. I'm not doing this for two reasons: one, distributions seem more complicated than is strictly necessary here; and two, it's impossible in practice to have a real point source, so if you work with distributions then you also need to understand what happens with bounded approximations to distributions, in which case you're back where you started. Of course you're welcome to work through this using distributions. End of digression.)

Define $\delta^h_{\theta_0} : S^2 \to \mathbb{R}$ by

$$\delta^h_{\theta_0}(\theta) = \begin{cases} h^{-2} & \text{if } |\theta - \theta_0| < h \\ 0 & \text{otherwise} \end{cases}$$

and $\delta^h_{x_0} : \partial X \to \mathbb{R}$ by

$$\delta^h_{x_0}(x) = \begin{cases} h^{-2} & \text{if } |x - x_0| < h \\ 0 & \text{otherwise} \end{cases}$$

Pick $(x_0, \theta_0) \in \Gamma_-$ and $h \ll 0$, and define $f = \delta^h_{x_0}(x)\delta^h_{\theta_0}(\theta)$. The boundary condition $f|_{\Gamma_-}$ corresponds to light restricted to a small neighbourhood of x, concentrated in direction θ_0. In other words, it's a laser pointer. Let's examine the solution to the RTE with boundary condition $f|_{\Gamma_-}$.

By (3.1), the solution looks like

$$u = (I + T^{-1}A_2 + T^{-1}A_2T^{-1}A_2 + \ldots)Jf.$$

If Jf is the nonscattered light, then it should look more or less like a laser pointer in air: it's zero everywhere unless you're both on the beam *and* looking in the direction of the laser pointer. Then it's unhealthily bright: don't look down the beam of the laser pointer!

You can see this just by writing down Jf: it's

$$Jf(x, \theta) = e^{-\tau(x, \gamma_-(x,\theta))} f(\gamma_-(x, \theta), \theta),$$

so only two things can happen: either $gamma_-(x, \theta)$ is in the h diameter neighbourhood of x_0 and θ is in the h diameter neighbourhood of θ, in which case Jf is $O(h^{-4})$, or at least one of those things fails, in which case we get zero.

Ok, good. Now look at the single scattering light. Intuitively it's light that changes direction exactly once. What should we expect here? All of the light was originally travelling down the beam from x_0 in the direction θ_0. This means that if you're looking at light that scattered exactly once, you must be looking *at the beam*: you must be seeing light that started in the beam, changed its mind once and then travelled down a straight line from the beam to you. You can stand anywhere, but you have to be looking at the beam.

If you think of what a laser pointer looks like in a little fog or mist, you'll see that this corresponds to something: in a scattering medium like mist, the beam itself becomes visible even if you're not looking directly down the barrel of the pointer. The single scattering light corresponds to the light that lets you see the beam. Notably, the beam itself has medium brightness: it's not nearly as bright as the concentrated laser fury you'd get if you stared directly down the pointer, but it still stands out against the general glow of the rest of the mist.

To sum up, the single scattered light is concentrated (it's supported for any x but only for θ such that the ray from x in the direction $-\theta$ intersects the original beam) but not as concentrated as the unscattered light, and it's bright (you can see the beam stand out against the background) but not as bright as the unscattered light can be.

What about the doubly-scattered light? If you want to see light that scattered twice, you can stand anywhere and look in any direction. It corresponds to some part of the generalized glow given off by the illuminated mist, so it must be substantially weaker than the singly scattered light. The doubly scattered light isn't particularly concentrated, and it isn't particularly bright. It's not clear that this light stands out from triply and higher order multiply scattered light.

I introduced this section by pitching it as a test of our understanding of our equation, but now that we're here, it really suggests a way to solve the inverse problem.

If you use an approximate point source f, then the discussion above suggests that on the support of Jf, Jf is really all you see. In particular, if you go to $\gamma_+(x_0, \theta_0)$, then you'll just measure Jf plus some lower order errors. And Jf is the solution to the non-scattering problem, so finding σ reduces to the X-ray transform just like it did before.

What about k? If you're not in the support of Jf, then the discussion above suggests that the main term is the single scattering light. That's just the scattering times the attenuation along the broken ray. So if you understand the total attenuation using Jf, then you should be able to recover the scattering from the single scattering term.

All we have to do is make this idea rigorous.

4. February 2

First we'll need some estimates.

4.1. Estimates.

Lemma 4.1. *Note that at any $x \in X$,*

$$\|A_2(w)(x,\cdot)\|_{L^\infty(S^{n-1})} \leq C_k \|w(x,\cdot)\|_{L^1(S^{n-1})}. \tag{4.1}$$

Moreover

$$\|T_1^{-1} w\|_{L^\infty(X\times S^{n-1})} \lesssim \|w\|_{L^\infty(X\times S^{n-1})}. \tag{4.2}$$

Proof. First, we have

$$A_2 w(x,\theta) = \int_{S^{n-1}} k(x,\theta,\theta') w(x,\theta')\, d\theta',$$

so

$$|A_2 w(x,\theta)| = \|k\|_{L^\infty(X\times S^{n-1}\times S^{n-1})} \int_{S^{n-1}} |w(x,\theta')|\, d\theta',$$

and the first result follows. Then

$$T^{-1} w(x,\theta) = \int_0^{|x-\gamma_-(x,\theta)|} e^{-\tau(x,x-t\theta)} w(x-t\theta,\theta)\, dt,$$

so

$$|T^{-1} w(x,\theta)| \leq \int_0^{|x-\gamma_-(x,\theta)|} e^{-\tau(x,x-t\theta)}\, dt \|w\|_{L^\infty(X\times S^{n-1})}.$$

The exponential inside the integral is bounded above by one, and the total length of the interval of integration is bounded by the diameter of X, so the second result follows too. □

4.2. **Recovering σ.** Now let's analyze our point source properly, with an eye to recovering σ. For convenience we'll stick to three dimensions for this discussion.

Recall that $f = \delta^h_{x_0}(x)\delta^h_{\theta_0}(\theta)$ where

$$\delta^h_{\theta_0}(\theta) = \begin{cases} h^{-2} & \text{if } |\theta - \theta_0| < h \\ 0 & \text{otherwise} \end{cases}$$

and

$$\delta^h_{x_0}(x) = \begin{cases} h^{-2} & \text{if } |x - x_0| < h \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$f(x,\theta) = \begin{cases} h^{-2} & \text{if } |x - x_0| < h \text{ and } |\theta - \theta_0| < h \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$Jf(x,\theta) = e^{-\tau(x,\gamma_-(x,\theta))} f(\gamma_-(x,\theta),\theta)$$

is $O(h^{-4})$ if $(\gamma_-(x,\theta),\theta)$ is in the support of f, and zero otherwise. In particular

$$Jf(\gamma_+(x_0,\theta_0),\theta_0) = h^{-4} e^{-\tau(\gamma_+(x_0,\theta_0),\gamma_-(x_0,\theta_0))}. \tag{4.3}$$

If we multiply by h^4, we have precisely the kind of thing we did in the X-ray (nonscattering) case: the exponential of the integral of σ along the line from x_0 in direction θ_0. We just need to show that the other terms don't interfere here.

So let's look at the next term in the collision expansion: $T^{-1}A_2 Jf(x,\theta)$. Note that for any fixed θ,

$$\|Jf(x,\cdot)\|_{L^1(S^{n-1})} \leq h^{-2} \int \delta^h_{\theta_0}(\theta)\, d\theta = O(h^{-2})$$

Then Lemma 4.1 implies that

$$A_2 Jf(x,\theta) \leq O(h^{-2})$$

for all $(x,\theta) \in X \times S^{n-1}$, and so

$$T^{-1}A_2 Jf(x,\theta) \leq O(h^{-2})$$

for all $(x,\theta) \in X \times S^{n-1}$. Therefore the next term in the collision expansion is small. What about the others? Well

$$\|(T^{-1}A_2 + (T^{-1}A_2)^2 + \ldots)Jf\|_{L^\infty(X\times S^{n-1}} \leq \frac{1}{1-c_1}\|T^{-1}A_2 Jf\|_{L^\infty(X\times S^{n-1}}$$

by the exact same summation of a geometric series that we used to prove the estimate in Theorem 3.1. Therefore if u is the solution to the RTE with boundary condition $u|_{\Gamma_-} = f|_{\Gamma_-}$, we have

$$u(x,\theta) = Jf(x,\theta) + O(h^{-2}).$$

In particular, equation (4.3) tells us that at the point $(\gamma_+(x_0,\theta_0),\theta_0) \in \Gamma_+)$, we have

$$u(\gamma_+(x_0,\theta_0),\theta_0) = h^{-4} e^{-\tau(\gamma_+(x_0,\theta_0),\gamma_-(x_0,\theta_0))} + O(h^{-2}).$$

Therefore

$$\lim_{h\to 0} h^4 u(\gamma_+(x_0,\theta_0),\theta_0) = e^{-\tau(\gamma_+(x_0,\theta_0),\gamma_-(x_0,\theta_0))}.$$

This reduces our problem to the X-Ray transform again! We can recover σ with the usual stability for that problem.

4.3. **Recovering k.** We won't do this in full but let's briefly sketch the argument. To recover k, you can keep the same point source that we had in the previous subsection, but look at a different spot on the boundary. Choose θ_1 such that $|\theta_1 - \theta_0| \gg h$, and pick x in the line $\{x_0 + t\theta_0\}$. Fix $x_1 = \gamma_+(x, \theta_1)$, and let's consider $u(x_1, \theta_1)$.

Because $|\theta_1 - \theta_0| \gg h$, $Jf(x_1, \theta_1) = 0$, and so the leading term is $T^{-1}A_2Jf$.

Using the explicit forms of the operators T^{-1} and A_2, together with the fact that δ_h approximates δ distributionsm, we find that

$$T^{-1}A_2Jf(x_1, \theta_1) = h^{-1}C_{\theta_1,\theta_0}k(x, \theta_1, \theta_0)e^{-\tau(x,\gamma_-(x,\theta_0))-\tau(x_1,x)} + o(1)$$

where x is the point of intersection of the lines $\{x_0+s\theta_0\}$ and $\{x_1+s\theta_1\}$, and C_{θ_1,θ_0} measures the length of the intersection of the line $\{x_1-t\theta_1\}$ with the support of $\delta^h_{x_0}(\gamma_-(x_1-t\theta_1, \theta_0))$. Now if we already know σ, then we can divide by the exponential at the end and recover $k(x, \theta_1, \theta_0)$.

The last thing to do is make sure that the remaining terms in the collision expansion are small.

Actually, because of a "sum the geometric series" argument like the one in the previous subsection, it really suffices to make sure that the next term in the collision expansion is small. This can be verified using Lemma 4.1: one ought to be able to show that the remainder is $O(1)$.