1. March 19

1.1. **Unique Continuation.** The term "unique continuation", like the term "inverse problem", is not super well defined and usually better illustrated by example than by a vague definition.

That said, unique continuation has a number of well-defined terms associated to it. The most well known are probably the following.

Definition 1.1. *We say the equation $\mathcal{L}u = 0$ on $\mathbb{R}^n$ has the weak unique continuation property if the condition*

$$\mathcal{L}u = 0 \text{ and } u \equiv 0 \text{ on an open set } U$$

implies that $u \equiv 0$ on $\mathbb{R}^n$.

Definition 1.2. *We say the equation $\mathcal{L}u = 0$ on $\mathbb{R}^n$ has the strong unique continuation property if the condition*

$$\mathcal{L}u = 0 \text{ and there exists } x_0 \in \mathbb{R}^n \text{ such that } \lim_{x \to x_0} |x - x_0|^{-m} u(x) = 0$$

for all $m \in \mathbb{N}$ implies that $u \equiv 0$ on $\mathbb{R}^n$.

Clearly the strong unique continuation property implies the weak unique continuation property.

We can modify these statements in the appropriate way to discuss unique continuation at infinity, too.

Note that the equation $\triangle u = 0$ has the strong unique continuation property, since any harmonic function is analytic.

Roughly speaking, unique continuation results tell you that knowing about a solution u in the neighbourhood of a point suffices to determine that solution globally. These are interesting in their own right, but they are also technical tools that crop up in a number of places, because they extend local information to global information.

As a cheap example, consider the following version of the standard Liouville theorem from complex analysis:

Theorem 1.1. *Suppose f is holomorphic and $\lim_{|z| \to \infty} f(z) = 0$. Then f is the zero function.*

If you like you can think of this as a (very strong) unique continuation result for holomorphic functions at infinity. The Liouville theorem gives rise to a very nice proof of the fundamental theorem of algebra.

Theorem 1.2. *Suppose $f(z)$ is a nonconstant polynomial. Then there exists a complex number z_0 such that $f(z_0) = 0$.*

Proof. you □

2. March 21

As an introduction to the subject, we'll prove the following simple unique continuation theorem at infinity.

Theorem 2.1. *Suppose $v \in H^2(\mathbb{R}^n)$ and there exists a constant C such that*

$$|\triangle v(x)| \leq C|v(x)|$$

for a.e. $x \in \mathbb{R}^n$. There exists $\tau > 0$ such that if

$$e^{\tau x^2} v \in H^2(\mathbb{R}^n)$$

Then $v \equiv 0$ on $\mathbb{R}^n$.

This immediately implies the weak unique continuation property at infinity for the Schrödinger equation $\triangle + q$ for L^∞ functions q. On closer inspection it proves something stronger than the weak unique continuation property, but not quite as strong as the strong unique continuation property. In practice, many unique continuation arguments take this form.

It's worth noting that unique continuation results like the one above apply to nonlinear equations as well. For example, if we have a solution to

$$\triangle u = u^2$$

that satisfies the inequality

$$|u(x)| \leq \frac{e^{-x^2}}{1 + |x|^{n+1}}$$

for all $x \in \mathbb{R}^n$, then the theorem above tells us that $u \equiv 0$.

2.1. **Carleman Estimate.** As a start, we will prove the following Carleman estimate. For my convenience, let's denote $\varphi(x) = |x|^2$.

Theorem 2.2. *Let $\tau > 0$, and define*

$$\triangle_\varphi = e^{\tau x^2} \triangle e^{-\tau x^2}.$$

There exists $C, \tau_0 > 0$ such that for all $u \in H^2(\mathbb{R}^n)$ and $\tau > \tau_0$

$$\tau \|u\|_{H^1(\mathbb{R}^n)} \leq C \|\triangle_\varphi u\|_{L^2(\mathbb{R}^n)}.$$

Proof. The proof is like the proof of our inverse problems Carleman estimate. Explicitly,

$$\triangle_\varphi u = e^{\tau\varphi} \triangle e^{-\tau\varphi} u = (\triangle - \tau(\nabla \cdot \nabla\varphi + \nabla\varphi \cdot \nabla) + \tau^2 |\nabla\varphi|^2) u.$$

Like in the inverse problems estimate, we notice that this can be written in terms of a self adjoint operator and an anti-self adjoint operator. If we define

$$A = \triangle + \tau^2 |\nabla\varphi|^2$$

and

$$B = -\tau(\nabla \cdot \nabla\varphi + \nabla\varphi \cdot \nabla)$$

then we have

$$\triangle_\varphi = A + B$$

where A is self adjoint and B is anti-self adjoint. We can write

$$\|\triangle_\varphi u\|^2 = \|Au\|^2 + \|Bu\|^2 + (Au, Bu) + (Bu, Au)$$

and integrate by parts. We get

$$\|\triangle_\varphi u\|^2 = \|Au\|^2 + \|Bu\|^2 + ([A,B]u,u)$$

with no boundary terms, since $u \in H^2$. In the inverse problems version of this estimate, the commutator term was zero, and the $\|Bu\|$ term gave us positivity because of Poincaré's inequality. Here, Poincaré alone cannot save us, because we are on an unbounded domain, so we'd better hope we have a good commutator. Explicitly,

$$[A,B] = 4\tau^3 \nabla\varphi \cdot D^2\varphi \cdot \nabla\varphi - 4\tau^3 \nabla \cdot D^2\varphi \cdot \nabla - \tau[\triangle, \triangle\varphi].$$

Our specific choice of φ is $\varphi(x) = |x|^2$, so the last term is zero. Moreover

$$\nabla\varphi(x) = 2x$$

and

$$D^2\varphi = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Therefore

$$[A,B] = 16\tau^3|x|^2 - 8\tau\triangle.$$

Therefore

$$([A,B]u,u) = 16\tau^3\|xu\|^2 + 8\tau\|\nabla u\|^2.$$

Then

$$\|\triangle_\varphi u\|^2 \geq 16\tau^3\|xu\|^2 + 8\tau\|\nabla u\|^2.$$

Now our idea should be that the first term bounds u when x is large and the second term bounds u when u is small. We can put this into practice by letting χ be a smooth radial cutoff function which is identically 1 when $|x| < 1$ and identically 0 when $|x| > 2$. Then

$$\begin{aligned}
8\tau\|\nabla u\|^2 &= 7\tau\|\nabla u\|^2 + \tau\|\chi\nabla u + (1-\chi)\nabla u\| \\
&\geq 7\tau\|\nabla u\|^2 + \tau\|\chi\nabla u\| - \tau\|(1-\chi)\nabla u\| \\
&\geq 6\tau\|\nabla u\|^2 + \tau\|\chi\nabla u\| \\
&\geq 6\tau\|\nabla u\|^2 + \tau\|\nabla(\chi u)\| - \tau\|(\nabla\chi)u\| \\
&\geq 6\tau\|\nabla u\|^2 + \tau\|\nabla(\chi u)\| - 2\tau\|xu\|
\end{aligned}$$

Therefore

$$\|\triangle_\varphi u\|^2 \geq 14\tau^3\|xu\|^2 + \tau\|\nabla(\chi u)\|^2 + 6\tau\|\nabla u\|^2.$$

Using a Poincaré inequality, we get

$$C\|\triangle_\varphi u\|^2 \geq 14\tau^3\|xu\|^2 + 6\tau\|\nabla u\|^2 + \tau\|\chi u\|^2 \geq \tau\|u\|^2_{H^1}.$$

□

Now let's prove the unique continuation result.

Proof of Theorem **??**. Suppose $e^{\tau x^2} v \in H^2(\mathbb{R}^n)$. Let $u = e^{\tau x^2} v$. Then $u \in H^2(\mathbb{R}^n)$, and by Theorem **??**,

$$\tau \|u\|_{L^2(\mathbb{R}^n)} \lesssim \|\triangle_\varphi u\|_{L^2(\mathbb{R}^n)}$$

for all sufficiently large τ. Now if

$$|\triangle v(x)| \leq C|v(x)|$$

for a.e. $x \in \mathbb{R}^n$, then

$$|\triangle_\varphi u| \leq C|u(x)|.$$

Therefore

$$\tau \|u\|_{H^1(\mathbb{R}^n)} \lesssim \|u\|_{L^2(\mathbb{R}^n)}$$

for all sufficiently large τ. But this is a contradiction unless $u \equiv 0$. □

This gives a toy example of a unique continuation result at infinity. Several inefficiencies in the proof should jump out at you: for one thing, it's not clear that $\varphi(x) = |x|^2$ is the optimal choice of Carleman weight.

For another thing, it should be evident to you that we can prove a unique continuation result with

$$(\triangle + W(x) \cdot \nabla) v(x) \leq C|v(x)|$$

with bounded W.

However, in the interest of broad coverage, I think it's time to move on.

What about unique continuation at a point? By translation, it suffices to consider what happens at zero without loss of generality.

Theorem 2.3. *Suppose that* $u \in H^2(\mathbb{R}^n)$ *and*

$$|\triangle u(x)| \leq C|u(x)|$$

for a.e. $x \in \mathbb{R}^n$*. There exists* $\tau_1 > 0$ *such that if*

$$|u(x)| \lesssim e^{-\frac{\tau_1}{x^2}}$$

in a neighbourhood of 0*, then* $u \equiv 0$*.*

Proof. The key idea is the behaviour of $\triangle$ under the Kelvin transform.

$$f^*(x) = \frac{1}{|x|^{n-2}} f\left(\frac{x}{|x|^2}\right)$$

One can check that

$$\triangle u^*(x) = \frac{1}{|x|^{n+2}} [\triangle u]\left(\frac{x}{|x|^2}\right).$$

In other words, if

$$|\triangle u(x)| \leq C|u(x)|$$

then

$$|\triangle u^*(x)| \leq \frac{C}{|x|^{n+2}} \left| u\left(\frac{x}{|x|^2}\right)\right| = \frac{C}{|x|^4} \left|u^*(x)\right|.$$

Moreover if

$$|u(x)| \lesssim e^{-\frac{\tau_1}{x^2}}$$

in a neighbourhood of zero, then

$$|u^*(x)| \lesssim e^{-\tau_1 x^2}$$

in a neighbourhood of infinity, and hence on $\mathbb{R}^n$. Therefore if u is bounded then u must be L^2, and by the first hypothesis in the theorem u must be H^2. Then $e^{\tau x^2} u \in H^2(\mathbb{R}^n)$ for any $\tau < \tau_1$, so for sufficiently large τ_1, we have, by Theorem **??**, that $u \equiv 0$. □

It's actually not necessary to stipulate that $|\triangle u(x)| \leq C|u(x)|$ hold in all of $\mathbb{R}^n$; it suffices if that happens in a neighbourhood of 0. If you believe that, here's a neat application of unique continuation.

Theorem 2.4. *Suppose Ω is a bounded domain with smooth boundary, $q \in L^\infty(\Omega)$ and $u \in H^2(\Omega)$ solves*

$$\triangle u + qu = 0$$

in Ω. Suppose moreover that $\Gamma \subset \partial\Omega$ is an open set with

$$u = \partial_\nu u = 0 \text{ on } \Gamma.$$

Then $u \equiv 0$ in Ω.

An immediate corollary is that any two solutions that are equal up to first order on an open subset of the boundary must be equal.

Proof. Consider an extended domain $\tilde{\Omega}$ such that $\Omega \subset \tilde{\Omega}$ and $\partial\Omega \setminus \Gamma \subset \partial\tilde{\Omega}$.

We can extend u to a function $\tilde{u}$ on $\tilde{\Omega}$ by the zero extension, and check that $\tilde{u} \in H^1(\tilde{\Omega})$ is a weak solution to the equation $\triangle\tilde{u} + q\tilde{u} = 0$ on $\tilde{\Omega}$.

By standard regularity theory, $\tilde{u}$ must be $H^2(\Omega)$, and satisfies the inequality

$$|\triangle u(x)| \leq C|u(x)|$$

for a.e. $x \in \tilde{\Omega}$. Moreover $\tilde{u}$ is identically zero in an open set in $\tilde{\Omega}$, so the weak unique continuation property says that $\tilde{u} \equiv 0$ on $\tilde{\Omega}$, and hence $u \equiv 0$ on Ω.

□