1. March 26

1.1. Control Theory. Like everything else, control theory is an ill-defined term, but we'll illustrate with an example.

Suppose Ω is a smooth bounded domain. Let $\Omega_T = \Omega \times (0,1)$, and consider the heat equation

$$(\partial_t - \triangle)u = 0 \text{ in } \Omega_T \\ u(x,0) = u_0(x) \text{ on } \Omega. \tag{1.1}$$

The question is, given a target function $u_1(x)$ defined on Ω, are there boundary values we can impose on $\partial_\Omega \times (0,1)$, such that the solution to (1.1) has the property that $u(x,1) = u_1(x)$?

In general the answer is no: because solutions to the heat equation are smooth, only certain u_1 are acceptable. A better way to phrase the question might be to first define a space like

$$X = \{v_1 \in L^2(\Omega) | v_1(x) = v(x,1), \text{ where } (\partial_t - \triangle)v = 0 \text{ in } \Omega \times (0,T), v_0(x) = v(x,0) \in L^2(\Omega)\}.$$

Then we can ask, for a given $v_1 \in X$, is there $f \in L^2(\partial_\Omega \times (0,T))$ such that the solution to (1.1) with boundary condition $u = g$ on $\partial_\Omega \times (0,T)$ satisfies $u(x,T) = v_1(x)$?

Notice that if $v_1 \in X$, then by definition there exists $v_0 \in L^2(\Omega)$ such that

$$(\partial_t - \triangle)v = 0 \text{ in } \Omega \times (0,1) \\ v(x,0) = v_0(x) \text{ on } \Omega$$

and $v(x,1) = v_1(x)$. Therefore if u solves (1.1) and $u(x,1) = v_1(x)$ then

$$(\partial_t - \triangle)(u - v) = 0 \text{ in } \Omega \times (0,1) \\ (u - v)(x,0) = (u_0 - v_0)(x) \text{ on } \Omega$$

and $u - v = 0$ at $t = 1$. The converse is true as well, so to answer the question of controllability, it suffices to understand the question of null-controllability.

1.2. Null-controllability. The basic question of null-controllability is the following: given a function u_0 on Ω, are there boundary conditions $g \in L^2(\partial\Omega \times (0,1))$ such that the equation

$$(\partial_t - \triangle)u = 0 \text{ in } \Omega \times (0,1) \\ u(x,0) = u_0(x) \text{ on } \Omega \\ u|_{\partial\Omega \times (0,1)} = g \tag{1.2}$$

has the property $u(x,1) \equiv 0$? If so, we say that the initial condition u_0 can be null controlled. More generally, we want to know what space E of functions defined on Ω has the property that each $u_0 \in E$ can be null controlled.

We say that a space E of functions on Ω can be null-controlled if there exists $C > 0$ such that for any $u_0 \in E$, there exists $g \in L^2(\partial\Omega \times (0,1))$ such that the solution u to the boundary value problem

$$\begin{cases} (\partial_t - \triangle)u &= 0 \\ u(0,x) &= u_0(x) \\ u|_{\partial\Omega\times(0,1)} &= g \end{cases} \tag{1.3}$$

satisfies the condition $u(1,x) \equiv 0$ *and*

$$\|g\|_{L^2(\partial\Omega\times(0,1))} \leq C\|u_0\|_E. \tag{1.4}$$

The last inequality is key to ensuring that the boundary conditions can be picked continuously, which is to say that the map from E to the necessary boundary condition is not too badly behaved.

To understand how on earth anyone solves a problem like this, we need to introduce the concept of the observability inequality.

1.3. **Observability Inequality.**

Theorem 1.1. *The following statements are equivalent:*

- *The space $L^2(\Omega)$ is null-controlled.*
- *For all $v \in H^2(\Omega_T)$ which solve $(\partial_t + \triangle)v = 0$ on Ω_T with $v \equiv 0$ on $\partial\Omega \times (0,1)$,*

$$\|v\|_{L^2(\Omega\times\{0\})} \lesssim \|\partial_\nu v\|_{L^2(\partial\Omega\times(0,1))}. \tag{1.5}$$

Proof. First, suppose $L^2(\Omega)$ is null-controlled, and suppose $v \in H^2(\Omega_T)$ solves $(\partial_t+\triangle)v = 0$ on Ω_T with $v \equiv 0$ on $\partial\Omega\times(0,1)$. Now in particular $v|_{t=0}$ is null-controlled, so we can set w to be a solution to $(\partial_t - \triangle)w = 0$ with initial condition $w(0,x) = v(0,x)$, final condition $w(1,x) \equiv 0$, and boundary condition $w|_{\partial\Omega\times(0,1)}$ satisfying (1.4). Then

$$\begin{aligned} 0 &= \int_{\Omega_T} w(\partial_t+\triangle)v\,dx \\ &= \int_{\Omega_T} (-\partial_t+\triangle)wv\,dx - \int_\Omega w(0,x)v(0,x)\,dx - \int_{\partial\Omega\times(0,1)} w\partial_\nu v\,dS \\ &= -\|v\|^2_{L^2(\Omega\times\{0\})} - \int_{\partial\Omega\times(0,1)} w\partial_\nu v\,dS. \end{aligned}$$

Therefore

$$\|v\|^2_{L^2(\Omega\times\{0\})} \leq C\|w\|_{L^2(\partial\Omega\times(0,1))}\|\partial_\nu v\|_{L^2(\partial\Omega\times(0,1))}.$$

By (1.4),

$$\|v\|_{L^2(\Omega\times\{0\})} \lesssim \|\partial_\nu v\|_{L^2(\partial\Omega\times(0,1))}.$$

Therefore (1.5) holds for each $v \in H^2(\Omega_T)$ which solves $(\partial_t+\triangle)v = 0$ on Ω_T with $v = 0$ on $\partial\Omega\times(0,1)$.

Now suppose (1.5) holds for each $v \in H^2(\Omega_T)$ which solves $(\partial_t+\triangle)v = 0$ on Ω_T with $v = 0$ on $\partial\Omega\times(0,1)$. Define

$$E = \{f \in L^2(\partial\Omega\times(0,1)) | f = \partial_\nu v \text{ for some } v \in H^2(\Omega_T) \text{ s.t. } (\partial_t+\triangle)v = 0 \text{ and } v|_{\partial\Omega\times(0,1)} = 0.\}$$

Let $u_0 \in L^2(\Omega)$. Define the linear functional $\varphi : E \to \mathbb{R}$ by

$$\varphi(f) = -(u_0, v|_{t=0})_\Omega.$$

Then (1.5) implies that

$$|\varphi(f)| \leq \|u_0\|_{L^2(\Omega)} \|v(0,x)\|_{L^2(\Omega)} \lesssim \|u_0\|_{L^2(\Omega)} \|f\|_{L^2(\partial\Omega \times (0,1))}.$$

Therefore φ is bounded as a map from $E \subset L^2(\partial\Omega \times (0,1))$ to $\mathbb{R}$, and by Hahn-Banach, it has an extension to the whole space $L^2(\partial\Omega \times (0,1))$.

The extension is a linear functional on an L^2 space, so by the Riesz representation theorem, there exists $g \in L^2(\partial\Omega \times (0,1))$ with

$$\|g\|_{L^2(\partial\Omega\times(0,1))} = \|\varphi\| \lesssim \|u_0\|_{L^2(\Omega)}$$

and

$$\varphi(f) = (f, g)_{\partial\Omega \times (0,1)}$$

for all $f \in L^2(\partial\Omega \times (0,1))$. Therefore

$$-(u_0, v|_{t=0})_\Omega = (f, g)_{\partial\Omega\times(0,1)}.$$

Now let u be the unique solution to $Pu = 0$ on Ω_T with initial condition $u(0,x) = u_0(x)$ and boundary conditions $u|_{\partial\Omega\times(0,1)} = g$. Then

$$-(u(0,x), v(0,x))_\Omega = (\partial_\nu v, u)_{\partial\Omega\times(0,1)}.$$

Meanwhile, an integration by parts gives

$$0 = ((-\partial_t + \triangle)u, v)_{\Omega_T} - (u, (\partial_t + \triangle)v)_{\Omega_T} = (u(1,x), v(1,x))_\Omega - (u(0,x), v(0,x))_\Omega - (\partial_\nu v, u)_{\partial\Omega\times(0,1)}.$$

Combining the above two equations gives

$$(u(1,x), v(1,x))_\Omega = 0.$$

This holds for any v solving the backwards heat equation $(\partial_t + \triangle)v = 0$ on Ω_T with $v \equiv 0$ on $\partial\Omega \times (0,1)$, but for any given $C^\infty(\Omega)$ function, we can solve the forwards heat equation on Ω_T and then time reverse the solution to obtain any smooth final condition for v that we want. Therefore $u(1,x) = 0$, and it follows that $L^2(\Omega)$ is null-controlled.

□

So the problem of control comes down to an inequality, like any other self-respecting problem in analysis. On the other hand, this inequality does not look very much like any Carleman inequality we've seen so far.

In fact, trying to prove a Carleman inequality for the (adjoint) heat equation, we might end up with the following, very different looking result.

Theorem 1.2. *Suppose without loss of generality that Ω does not contain* 0 *and is contained in the ball of radius R around* 0*. Let*

$$\varphi(x,t) = \frac{x^2 - R^2}{t(1-t)}.$$

Let

$$P_\varphi^* = e^{\tau\varphi}(\partial_t + \triangle)e^{-\tau ph}.$$

Then for all $u \in H^2(\Omega_T)$ *with* $u \equiv 0$ *on* $\partial\Omega_T$,

$$\tau\|u\|_{L^2(\Omega_T)} \lesssim \|\sqrt{|\partial_\nu\varphi|}\partial_\nu u\|_{L^2(\partial\Omega\times(0,1))} + \|P^*_\varphi u\|_{L^2(\Omega_T)}.$$

This has an extra boundary term on the right side, compared to the Carleman estimates we're used to. But this isn't so strange – if we're not assuming that v vanishes to first order on the boundary, we should expect to pick up a boundary term from the integration by parts that we usually use to prove Carleman estimates. What's the relationship between the Carleman estimate and the observability inequality? I claim the Carleman estimate in fact proves the observability inequality, and hence controllability. Let's see why.

First, we have to make the standard substitution $v = e^{-\tau\varphi}u$, so we get

$$\tau\|e^{\tau\varphi}v\|_{L^2(\Omega_T)} \lesssim \|\sqrt{|\partial_\nu\varphi|}e^{\tau\varphi}\partial_\nu v\|_{L^2(\partial\Omega\times(0,1))} + \|e^{\tau\varphi}(\partial_t + \triangle)v\|_{L^2(\Omega_T)}$$

for all v such that $e^{\tau\varphi}v \in H^2(\Omega_T)$ with $v \equiv 0$ on $\partial\Omega \times (0,1)$. Now $e^{\tau\varphi}$ and $\sqrt{|\partial_\nu\varphi|}e^{\tau\varphi}$is bounded above on Ω_T, so in fact

$$\|e^{\tau\varphi}v\|_{L^2(\Omega_T)} \lesssim \|\partial_\nu v\|_{L^2(\partial\Omega\times(0,1))} + \|(\partial_t + \triangle)v\|_{L^2(\Omega_T)}$$

for all $v \in H^2(\Omega_T)$ with $v \equiv 0$ on $\partial\Omega \times (0,1)$. (Note that we don't have to specify that v is zero on the time boundaries $t = 0$ or $t = 1$, because of the form of φ and the fact that $v = e^{-\tau\varphi}u$. Sadly $e^{\tau\varphi}$ is not bounded below on Ω_T. (why not?) But it is bounded below on $\Omega \times (1/3, 2/3)$, so

$$\|v\|_{L^2(\Omega\times(1/3,2/3))} \lesssim \|\partial_\nu v\|_{L^2(\partial\Omega\times(0,1))} + \|(\partial_t + \triangle)v\|_{L^2(\Omega_T)}.$$

Now what? In the observability inequality, we make the crucial assumption that $(\partial_t + \triangle)v = 0$. If we apply this, we get

$$\|v\|_{L^2(\Omega\times(1/3,2/3))} \lesssim \|\partial_\nu v\|_{L^2(\partial\Omega\times(0,1))}$$

for all $v \in H^2(\Omega_T)$ with $v \equiv 0$ on $\partial\Omega \times (0,1)$ and $(\partial_t + \triangle)v = 0$. This is almost the observability inequality! We only have the wrong lower bound. But a simple energy argument fixes this: if $(\partial_t + \triangle)v = 0$, then

$$\begin{aligned}
\partial_t\|v(x,t)\|^2_{L^2(\Omega)} &= 2\int_\Omega v(x,t)\partial_t v(x,t)\,dx \\
&= -2\int_\Omega v(x,t)\triangle v(x,t)\,dx \\
&= 2\|\nabla v(x,t)\|^2_{L^2(\Omega)}.
\end{aligned}$$

Therefore $\|v(x,t)\|^2_{L^2(\Omega)}$ is an increasing function of t, and

$$\|v\|_{L^2(\Omega\times\{0\})} \lesssim \|v\|_{L^2(\Omega\times(1/3,2/3))}.$$

Therefore

$$\|v\|_{L^2(\Omega\times\{0\})} \lesssim \|\partial_\nu v\|_{L^2(\partial\Omega\times(0,1))}$$

for all $v \in H^2(\Omega_T)$ with $v \equiv 0$ on $\partial\Omega \times (0,1)$ and $(\partial_t + \triangle)v = 0$. This proves the observability inequality, and hence the following theorem

Theorem 1.3. *Let Ω be a smooth bounded domain. Then the space $L^2(\Omega)$ is null-controlled for the heat equation.*

It only remains to prove the Carleman inequality.

Proof of the Carleman inequality. As usual, we begin by writing out in full

$$P_\varphi^* = \partial_t + \tau\partial_t\varphi + \triangle - \tau(\nabla \cdot \nabla\varphi + \nabla\varphi \cdot \nabla) + \tau^2|\nabla\varphi|^2.$$

We write this in terms of symmetric and antisymmetric parts:

$$P_\varphi^* = A + B$$

where

$$A = \triangle + \tau^2|\nabla\varphi|^2 + \tau\partial_t\varphi$$

and

$$B = \partial_t - \tau(\nabla \cdot \nabla\varphi + \nabla\varphi \cdot \nabla),$$

and we get

$$\|P_\varphi^* v\|^2_{L^2(\Omega_T)} = \|Au\| + \|Bu\| + ([A, B]u, u) + \text{ boundary terms.}$$

Because of the fact that u is zero on the boundary, only one boundary term in fact remains: it's the boundary term that you get from integrating by parts in the term

$$(2\tau\nabla\varphi \cdot \nabla u, \triangle u)_{\Omega_T} = 2\tau(\nabla\varphi \cdot \nabla u, \partial_\nu u)_{\partial\Omega\times(0,1)} + (\triangle 2\tau\nabla\varphi \cdot \nabla u, u)_{\Omega_T}$$

The tangential part of $\nabla\varphi \cdot \nabla u$ is also irrelevant, because of the fact that u vanishes on the boundary, so in fact what we're left with is the boundary term

$$2\tau(\partial_\nu\varphi\partial_\nu u, \partial_\nu u)_{\partial\Omega\times(0,1)}.$$

This is bounded above by

$$C\tau\|\partial_\nu u\|^2_{L^2(\partial\Omega\times(0,1))}.$$

The commutator is bounded below as before (there are some extra terms, but disposing of them is merely an exercise in tedium). This finishes the proof of the Carleman estimate.

□