1. March 5

1.1. **Acousto-Optics.** Whereas photoacoustics uses an optical wave to provide a source for an acoustic wave, acousto-optics uses an acoustic wave to interfere with an existing optical wave.

To get a sense of how this works, consider a optical scattering problem

$$\nabla\gamma \cdot \nabla u = 0 \tag{1.1}$$

where the scattering coefficient γ is physically determined by particles floating in an otherwise optically constant medium.

If we set up a known acoustic pressure wave of the form $1 + a\cos(kt)\cos(q \cdot x)$ inside the medium, then the particles get squeezed together in areas of high pressure and rarefied in areas of low pressure.

Let's pick a particular time t_0 to make an observation, so we have pressure

$$1 + ab\cos(q \cdot x)$$

for some b. If we assume γ is proportional to the pressure, then we get a new coefficient

$$\gamma_q(x) = (1 + abc\cos(q \cdot x))\gamma(x).$$

By folding together the coefficients $\varepsilon = abc$, we can write

$$\gamma_q(x) = (1 + \varepsilon\cos(q \cdot x))\gamma(x).$$

We will assume that the amplitude of the acoustic wave is small, (there's not much physical displacement due to the acoustic wave), so $\varepsilon \ll 1$. Under the influence of the acoustic wave, we get a new equation for light intensity

$$\nabla \cdot \gamma_q \nabla u_q = 0. \tag{1.2}$$

In general the solution u_q to this equation is not identical to the solution u to the unperturbed equation (1.1). The conclusion is that the acoustic wave has modulated the optical wave! This is sometimes called the acousto-optic effect.

Now suppose we can vary q and measure $u_q, \partial_\nu u_q$ for all q. Let's try to recover γ.

1.2. **Internal Functional.** As in the photoacoustic problem, we will break this down into two parts. First we will recover an internal functional, and then we'll try to see what we can do with the internal functional.

To recover the internal functional, we'll need to integrate something by parts.

Consider a solution u to

$$0 = \int_\Omega \nabla \cdot \gamma \nabla u u_q \, dx.$$

Integrating by parts gives

$$\int_\Omega \gamma \nabla u \nabla u_q \, dx = \int_{\partial\Omega} \gamma \partial_\nu u u_q \, dS.$$

Meanwhile, a second integration by parts tells us that

$$\int_\Omega \gamma_q \nabla u \nabla u_q \, dx = \int_{\partial\Omega} \gamma_q \partial_\nu u_q u \, dS.$$

The boundary terms are known, so we can recover the difference

$$\int_\Omega (\gamma - \gamma_q)\nabla u \cdot \nabla u_q \, dx. \tag{1.3}$$

This is still nonlinear in q, which is slightly annoying. Let's try to understand the difference between u and u_q. We have

$$\nabla \cdot \gamma_q \nabla u_q = 0,$$

and

$$\nabla \cdot \gamma \nabla u = 0$$

which is to say that

$$\nabla \cdot \gamma_q \nabla u = \varepsilon \cos(q \cdot x)\gamma \triangle u + \varepsilon \nabla(\gamma \cos(q \cdot x)) \cdot \nabla u$$

Therefore

$$\nabla \cdot \gamma_q \nabla (u - u_q) = \varepsilon \cos(q \cdot x)\gamma \triangle u + \varepsilon \nabla(\gamma \cos(q \cdot x)) \cdot \nabla u.$$

Assume $u = u_q$ on $\partial\Omega$. Then integrating by parts, we find

$$\int_\Omega \gamma_q \nabla(u - u_q) \cdot \nabla(u - u_q) dx = O(\varepsilon).$$

Since $\gamma - \gamma_q$ is also $O(\varepsilon)$,

$$\int_\Omega \gamma \nabla(u - u_q) \cdot \nabla(u - u_q) dx = O(\varepsilon).$$

Returning to (1.3), we have

$$\begin{aligned}
\int_\Omega (\gamma - \gamma_q)\nabla u \cdot \nabla u_q \, dx &= \int_\Omega (\gamma - \gamma_q)\nabla u \cdot \nabla u \, dx + \int_\Omega (\gamma - \gamma_q)\nabla u \cdot \nabla (u_q - u) \, dx \\
&= \int_\Omega (\gamma - \gamma_q)\nabla u \cdot \nabla u \, dx + O(\varepsilon^2). \\
&= \varepsilon \int_\Omega \cos(q \cdot x)\gamma \nabla u \cdot \nabla u \, dx + O(\varepsilon^2).
\end{aligned}$$

By dividing by ε and varying the ultrasound modulation q, we get the Fourier transform of

$$H(x) = \gamma |\nabla u|^2$$

up to $O(\varepsilon)$. This is the internal functional.

2. March 7

2.1. **Using the internal functional.** On one hand, this is much better than we did before: instead of having an integral of $\gamma|\nabla u|^2$, we have the function itself.

On the other hand, while it's known that γ can be recovered from the functional $H(x) = \gamma|\nabla u|^2$, there aren't really good ways of doing it.

The best known technique (as far as I'm aware) is described by Guillaume Bal: it's based on the observation that

$$\nabla \cdot \frac{H(x)}{|\nabla u|^2} \nabla u = 0.$$

This is a nonlinear equation – actually a p-Laplace equation with $p = 0$ and (known) coefficient H. Bal's method relies on analysis of the 0-Laplace with a known coefficient H to recover u and hence γ.

I suspect there must be a better method but as far as I know this is a bit of an open problem.

The situation in the RTE case is much better.

2.2. **RTE Acousto-optics.** Recall that for low-scattering media, we have a specific light intensity $u : \Omega \times S^2 \mapsto \mathbb{R}$ which is governed by the equation

$$\theta \cdot \nabla u + \sigma u = \int_{S^2} k(x, \theta, \theta') u(x, \theta') d\theta'. \tag{2.1}$$

For simplicity let's consider the case with *no* scattering: $k \equiv 0$.

If we assume thatσ is modulated by the acoustic wave, with

$$\sigma_q = (1 + \varepsilon \cos(q\dot{x}))\sigma,$$

we get

$$\theta \cdot \nabla u_q + \sigma_q u_q = 0 \tag{2.2}$$

In analogy to the elliptic case, we'll assume that we can set $u_q = f$ on Γ_-, and we can measure u_q on Γ_+ for any modulation q.

Again we can try to determine an internal functional by integration by parts. There's a small problem in this case: the RTE is not based on a self adjoint operator.

Therefore we need to consider solutions v to the adjoint RTE

$$-\theta \cdot \nabla v + \sigma v = 0 \tag{2.3}$$

What guarantees that solutions to this equation exist? It is straightforward to check that if u satisfies the regular RTE (2.1) with boundary condition $u|_{\Gamma_-} = f$, then the function $v(x, \theta) = u(x, -\theta)$ satisfies the adjoint RTE (2.3) with boundary condition $v|_{\Gamma_+} = f(x, -\theta)$, and vice versa. Therefore existence and uniqueness for the adjoint RTE is guaranteed by the usual existence and uniqueness for the regular RTE.

So suppose u_q satisfies the equation (2.2) with a boundary condition $u_q = f$ on Γ_- picked by us, and v satisfies the adjoint equation (2.3) with a boundary condition $v = g$ on Γ_+ picked by us. Then

$$\int_\Omega \theta \cdot \nabla u_q v \, dx = -\int_\Omega u_q \theta \cdot \nabla v \, dx + \int_{\partial\Omega} u_q v \, \theta \cdot \nu \, dS.$$

The boundary term is known, so we get

$$\int_\Omega \sigma_q u_q v \, dx - \int_\Omega u_q \sigma v \, dx = \text{ known } .$$

Therefore

$$\int_{S^2}\int_\Omega (\sigma_q - \sigma)u_q v\, dx d\theta. = \text{ known } .$$

As in the elliptic case, this is nonlinear in q, and we want to solve the problem by casting off $O(\varepsilon^2)$ errors. To this end, define a function u to solve the unmodulated RTE (2.1) with $u = u_q$ on Γ_-. Then

$$\theta \cdot \nabla u = \sigma_q u - \varepsilon \cos(q \cdot x)\sigma u,$$

and

$$\theta \cdot \nabla(u - u_q) = \sigma_q(u - u_q) - \varepsilon \cos(q \cdot x)\sigma u$$

with $u - u_q = 0$ on $\partial\Omega$. Solving this RTE for $u - u_q$ gives

$$\|u - u_q\|_{L^\infty(\Omega\times S^2)} = O(\varepsilon).$$

Therefore

$$\begin{aligned}\int_\Omega (\sigma_q - \sigma)u_q v\, dx\, d\theta &= \int_\Omega (\sigma_q - \sigma)uv\, dx\, d\theta + O(\varepsilon^2)\\ &= \varepsilon \int_\Omega \cos(q\cdot x)\sigma uv\, dx + O(\varepsilon^2).\end{aligned}$$

Just as in the elliptic case we recover, up to $O(\varepsilon)$ error, an internal functional

$$H(x) = \sigma uv$$

3. March 9

3.1. **Using the internal functional.** Now we need to recover σ from the functional. In principle this is difficult because both u and v depend on σ. But miraculously uv does not: one can check that

$$\theta \cdot \nabla(uv) = \sigma uv - \sigma uv = 0.$$

This implies that $u(x + t\theta, \theta)v(x + t\theta, \theta)$ is independent of t. Therefore the value of each $u(x, \theta)v(x, \theta)$ is equal to its value at some boundary point, and thus $u(x, \theta)v(x, \theta)$ is known everywhere inside Ω.

Contemplation of the solution operator should convince you that with the choice of positive boundary values, uv can be made positive everywhere inside Ω. This implies that we can simply divide through by uv to get σ.

Note that unlike the standard X-ray transform problem, stability does not depend on the derivative of anything: there's an algebraic formula for the recovery of σ from a functional, which is itself recovered by a Fourier transform.

3.2. **Inverse Source Problems.** A true advantage of the acousto-optic effect is that it can be exploited to make insoluble problems solvable.

A great example of this is the following inverse source problem.

Suppose $\triangle u = f$ inside Ω. We can measure $u, \partial_\nu u$ at $\partial\Omega$, and we want to recover f.

A moment's reflection should convince you that this is impossible: it is easy to find two functions u and v that match to first order at the boundary, but $\triangle u \neq \triangle v$.

On the other hand, if the source function f is subject to an acousto-optic effect, then the introduction of an acoustic wave gives us a new equation

$$\triangle u_q = f_q$$

where $f_q = (1 + \varepsilon \cos(q \cdot x))f(x)$. Integrating by parts,

$$\int_{\partial\Omega} \partial_\nu u \, dS = \int_\Omega f \, dx$$

and

$$\int_{\partial\Omega} \partial_\nu u_q \, dS = \int_\Omega f_q \, dx,$$

so subtracting,

$$\varepsilon \int_\Omega \cos(q \cdot x) f(x) \, dx = \int_{\partial\Omega} \partial_\nu (u_q - u) \, dS.$$

The right side can be measured and the left gives the Fourier transform of f.