Preliminary Examination in Numerical Analysis

June 2, 2006

Instructions:

- 1. The examination is for 3 hours.
- 2. The examination consists of two parts: Part I: Matrix Theory and Numerical Linear Algebra Part II: Introductory Numerical Analysis
- 3. There are three problem sets in each part. Work two out of the three problem sets for each part.
- 4. All problem sets carry equal weights.

PART I - Matrix Theory and Numerical Linear Algebra (Work two of the three problem sets in this part)

Problem 1.

1. Assume that A and $A + \delta A$ are $n \times n$ invertible matrices and $\eta := \kappa(A) \frac{\|\delta A\|}{\|A\|} < 1$. Prove that

$$\frac{\|(A+\delta A)^{-1}-A^{-1}\|}{\|A^{-1}\|} \le \frac{\kappa(A)\frac{\|\delta A\|}{\|A\|}}{1-\eta}$$

where $\|\cdot\|$ is any matrix operator norm and $\kappa(A)$ is the condition number of A.

- 2. Let $A \in \mathbb{R}^{n \times n}$ be symmetric positive definite. Prove by induction on n that A has a Cholesky factorization, namely there exists a lower triangular matrix G such that $A = GG^T$.
- 3. Write down the algorithm for computing the Cholesky factorization. Use the error analysis of the LU factorization to explain briefly why the Cholesky algorithm is stable without pivoting.

Problem 2. Let A be an $m \times n$ matrix and m > n.

- 1. Describe the Gram-Schmidt process to orthogonalize the columns of A. Prove that the process produces an orthonormal basis if the columns of A are linearly independent. Formulate the process into the matrix factorization A = QR, and what is Q and what is R?
- 2. Describe an algorithm for computing the QR factorization of A by the Householder transformations. How does this algorithm compare with the one based on the Gram-Schmidt process?

Problem 3.

- 1. State the power method for computing the largest eigenvalue (in absolute value) of a matrix $A \in \mathbb{R}^{n \times n}$. State and prove the convergence result for the case that A is symmetric. Describe the shift-and-invert method to compute the eigenvalue closest to a given number μ .
- 2. Let β be an approximate eigenvalue and $x \in \mathbb{R}^n$ with $||x||_2 = 1$ a corresponding approximate eigenvector of an $n \times n$ real symmetric matrix A. Prove

$$\min_{i} |\lambda_i - \beta| \le ||Ax - \beta x||_2.$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of A.

Part II – Numerical Analysis (Work two of the three problem sets in this part)

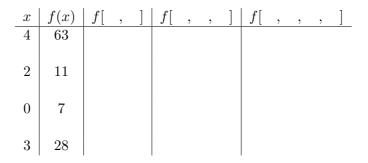
Problem 4. Polynomial interpolation.

1. Prove that if g interpolates the function f at $x_0, x_1, \ldots, x_{n-1}$ and if h interpolates f at x_1, x_2, \ldots, x_n , then the function

$$g(x) + \frac{x_0 - x}{x_n - x_0}[g(x) - h(x)]$$

interpolates f at $x_0, x_1, \ldots, x_{n-1}, x_n$.

2. Complete the following divided-difference table



- 3. Obtain polynomials of degree 3 that interpolates the function values given in (2), using Newton's formula.
- 4. Obtain polynomials of degree 3 that interpolates the function values given in (2), using Lagrange's formula. (You may just write down the formula without simplifying it.)

Problem 5. Quadrature rules.

- 1. Given (n + 1) nodes x_0, x_1, \ldots, x_n in [a, b]. Derive the Newton-Cotes formula for approximating $\int_a^b f(x) dx$. Write down the trapezoid rule (for which $n = 1, x_0 = a$, and $x_1 = b$).
- 2. Prove that the Newton-Cotes formula you just derived is exact for all polynomials of degree no higher than n.
- 3. Find the Gaussian quadrature rule

$$\int_{-1}^{1} f(x) \, dx \approx A_0 f(x_0) + A_1 f(x_1) + A_2 f(x_2).$$

What is the highest degree of polynomials for which this formula is exact?

4. Use the Gaussian quadrature rule you just found to find the Gaussian quadrature rule

$$\int_{a}^{b} g(t) dt \approx A_0 f(t_0) + A_1 f(t_1) + A_2 f(t_2).$$

Use it to approximate $\int_0^2 \frac{\sin t}{t} dt$.

Problem 6. Zeros of functions and splines.

- 1. What are the conditions that guarantee the bisection to work, i.e., to find a zero? Justify your answer.
- 2. What are the conditions that guarantee the Newton method to converge quadratically at the end? Justify your answer.
- 3. Given (n+1) knots $t_0 < t_1 < \cdots < t_n$, what are the conditions to make a function S(x) on $[t_0, t_n]$ a natural cubic spline.
- 4. Find the natural cubic spline that interpolates the table