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Preliminary Examination
Partial Differential Equations
May 2004

Instructions

This is a three-hour examination. You need to solve a total of **five problems**. The exam is divided into two parts. **You must do at least two problems from each part**. Please indicate clearly on your test papers which five problems are to be graded. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

PART ONE

1. Let $B = \{x : |x| < 1\} \subset \mathbb{R}^n$. Suppose u is twice continuously differentiable on B, continuous on the closure of B, $\Delta u = -1$ in B, and $u \geq 0$ on ∂B. Show that

$$\frac{1}{2n}(1 - |x|^2) \leq u(x) \leq \max\{u(y) : |y| = 1\} + \frac{1}{2n}(1 - |x|^2)$$

for all $x \in B$ (as usual Δu denotes the Laplacian of u).

2. Given $u(x,t) = x\, t^{-3/2} \exp[-x^2/4t]$ when $(x,t) \in \{(y,s) \in \mathbb{R}^2 : y > 0, s > 0\}$.

   (a) Prove that $u_t = u_{xx}$ when $x > 0$, $t > 0$.

   (b) Find $\lim_{x \to 0} u(x,t)$ when $t > 0$ and $\lim_{t \to 0} u(x,t)$ when $x > 0$.

   (c) Does the boundary value problem

$$\begin{aligned} &v_t = v_{xx} \text{ when } x > 0, t > 0, \\ &v(x,0) = 0 \text{ when } x > 0, t = 0 \\ &v(0,t) = 0 \text{ when } t > 0, x = 0 \end{aligned}$$

   have a unique solution?

3. Use the method of characteristics to solve (locally) the first order Hamilton - Jacobi partial differential equation:

$$u_{x_1} u_{x_2} = u \text{ in } V = \{(x_1, x_2) : x_1 \in (0,\infty),\, x_2 \in \mathbb{R}\}$$

$$u(0, x_2) = x_2^2 \text{ on } \partial V\,.$$

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4. Let $U = \{(x,t) : x \in \mathbb{R}^3, t \in (0,\infty)\}$ and suppose that $u \in C^2(U) \cap C^1(\bar{U})$ is a solution to the following wave equation:

$$u_{tt} - \Delta u = f \text{ in } U$$

$$u(x,0) = \phi(x) \text{ for } x \in \mathbb{R}^3$$

$$u_t(x,0) = \psi(x) \text{ for } x \in \mathbb{R}^3.$$

Here $f \in C_0^\infty(U)$ and ϕ, $\psi \in C_0^\infty(\mathbb{R}^3)$. Assuming ' appropriate decay of u on time slices in a neighborhood of ∞, ' use the energy method to derive the following estimate:

$$E(t)^{1/2} \le E(0)^{1/2} + t^{1/2}\left(\int_{\mathbb{R}^3\times[0,t]} |f(x,s)|^2 dx ds\right)^{1/2}.$$

where if $\nabla u = (u_{x_1}, u_{x_2}, u_{x_3})$, then

$$E(t) = \int_{\mathbb{R}^3} (|\nabla u|^2 + |u_t|^2)(x,t) dx$$

and

$$E(0) = \int_{\mathbb{R}^3} (|\nabla \phi|^2 + |\psi|^2)(x) dx$$

PART TWO

1. Let $B_r = \{y \in \mathbb{R}^n : |y| < r\}$ and suppose that $u \in W^{1,1}(B_r) \cap C^1(B_r)$.

(a) Show there exists, c, $1 \le c < \infty$, such that

$$|u(x) - u_{B_r}| \le c \int_{B_r} \frac{|\nabla u|(y)}{|x-y|^{n-1}} dy, \text{ whenever } x \in B_r.$$

Here $u_{B_r} = |B_r|^{-1} \int_{B_r} u\, dy$ is the average of u over B_r and $\nabla u = (u_{x_1}, \dots, u_{x_n})$.

(b) Given that smooth functions in B_r are dense in $W^{1,1}(B_r)$. Use (a) to prove Poincare's inequality for B_r and $p = 1$: If $v \in W^{1,1}(B_r)$, then for some c', $1 \le c' < \infty$,

$$\|v - v_{B_r}\|_{L^1(B_r)} \le c' r \, \|\nabla v\|_{L^1(B_r)}.$$

(c) Prove or disprove: There exists, $\tilde{c}, 1 \le \tilde{c} < \infty$, such that if $v \in W^{1,1}(B_r)$, then

$$\|v\|_{L^1(B_r)} \le \tilde{c} r \, \|\nabla v\|_{L^1(B_r)}.$$

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2. Suppose $u \in H^1(\Omega)$ where Ω is a bounded open set in $\mathbb{R}^n$. Show that

$$\int_\Omega Du \cdot D\varphi \, dx = 0$$

for each $\varphi \in C_0^\infty(\Omega)$ if and only if,

$$\int_\Omega |Du|^2 \, dx \le \int_\Omega |Dv|^2 \, dx$$

for each $v \in H^1(\Omega)$ such that $u - v \in H_0^1(\Omega)$.

3. Given $f \in L^2(\mathbb{R}^n)$, $u \in H^1(\mathbb{R}^n)$ with compact support, $b \in C^1(-\infty,\infty)$ with $b(0) = 0$, and $b' \ge 0$ on $(-\infty,\infty)$. Also suppose that u is a weak solution on $\mathbb{R}^n$ to

$$Lu(x) = -\Delta u(x) + b(u(x)) = f(x), \; x \in \mathbb{R}^n.$$

Use the method of difference quotients to prove that $u \in H^2(\mathbb{R}^n)$.

4. Let $(a_{ij}(x), 1 \le i,j \le n)$ be an n by n symmetric matrix with coefficients in $C^1(\bar{U})$, where $U \subset \mathbb{R}^n$ is a bounded open set.

   (a) What is meant by the phrase, (a_{ij}), are uniformly elliptic on U?

   (b) Show that if $w \in C^2(U) \cap C(\bar{U})$ is a solution to

   $$Lw(x) = \sum_{i,j=1}^{n} a_{ij}(x)\, w_{x_i x_j}(x) = 0 \text{ for all } x \in U,$$

   where (a_{ij}) are uniformly elliptic on U, then $\max_{\bar{U}} w = \max_{\partial U} w$.