Preliminary Examination in Partial Differential Equations
June 4, 2009

Instructions This is a three-hour examination.The exam is divided into two parts. You need to solve a total of five problems. You must do at least two problems from each part. Please indicate clearly on your test papers which five problems are to be graded.

You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

PART I

(1) Let $\{u_k\}$ be a sequence of harmonic functions in a smooth domain Ω. Suppose that $\{u_k\}$ converges to a harmonic function u uniformly on every compact subset of Ω. Show that $\{\nabla u_k\}$ also converges to $\{\nabla u\}$ uniformly on every compact subset of Ω.

(2) Suppose that $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ where Ω is a bounded smooth domain. Show that if $\Delta u = 0$ in Ω, then

$$\int_{\partial\Omega} |\nabla u|^2 < x, \nu > dS = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} (2 < \nabla u, x > + (n-2)u) dS,$$

where $x = (x_1, \ldots, x_n)$, ν is the outer unit normal to Ω and dS is surface area on $\partial\Omega$.

(3) Let $u \in C^2(\mathbf{R} \times [0, \infty))$ solve the initial value problem for the heat equation in one dimension:

$$\begin{aligned} u_t - u_{xx} &= \lambda(t) u \text{ in } \mathbf{R} \times (0, \infty) \\ u &= g \quad \text{ on } \mathbf{R} \times \{t = 0\}. \end{aligned}$$

If

$$\lambda(t) = \frac{\int_{\mathbf{R}} u_x^2(x,t)dx}{\int_{\mathbf{R}} g^2(x)dx}$$

prove that $I(t) = \int_{\mathbf{R}} u^2(x,t)dx$ is constant for $t > 0$.
You may assume that u decays fast enough on each time slice so that integration by parts can be used.

(4) Use the characteristic method to find an explicit solution to the first order partial differential equation:

$$\begin{aligned} u_{x_1} + u_{x_2} &= u^2 \text{ in } \mathbf{R}^2_+ \\ u &= g \quad \text{on } \partial\mathbf{R}^2_+ \end{aligned}$$

where $\mathbf{R}^2_+$ is the upper half plane, and g is a continuous function.

PART II

(5) Let $u \in W^{1,p}(\mathbf{R}^n)$ for some $p, 1 \le p < \infty$. If $y \in \mathbf{R}^n$ and $r > 0$ set $B(y,r) = \{y : |y - x| < r\}$. Given that for almost every $x \in B(y,r)$,

$$(*) \quad |u(x) - u_{B(y,r)}| \le c \int_{B(y,r)} \frac{|Du|(z)}{|x-z|^{n-1}}\, dz,$$

where $c = c(n)$, and $u_{B(y,r)}$ is the average of u on $B(y,r)$.

(a) If $p > n$ use $(*)$ to prove Morrey's theorem:
For almost every $x, y \in \mathbf{R}^n$,

$$|u(x) - u(y)| \le c'|x-y|^{1-n/p}\, \|Du\|_{L^p(\mathbf{R}^n)},$$

where $c' = c'(p,n)$.

(b) Use $(*)$ to prove Poincaré's inequality for $B(y,r)$ when $p = 1$:

$$\int_{B(y,r)} |u(x) - u_{B(y,r)}|\, dx \le \tilde{c}\, r \int_{B(y,r)} |Du(x)| dx$$

where $\tilde{c}$ depends only on n.

(6) Let $u \in L^2(\mathbf{R}^n)$ and let e_k be the point on the positive k th coordinate axis, $1 \le k \le n$, with $|e_k| = 1$. Set

$$D_k^h u(x) = \frac{u(x + he_k) - u(x)}{h}$$

whenever $x \in \mathbf{R}^n, 0 < h \le 1$, and $1 \le k \le n$. Prove that $u \in W^{1,2}(\mathbf{R}^n)$ if and only if $\|D_k^h u\|_{L^2(\mathbf{R}^n)} < \infty$ for $0 < h \le 1$ and $1 \le k \le n$.

(7) Let $U \subset \mathbf{R}^n$ be a bounded domain and let $u \in W^{1,2}(U)$ be a bounded weak solution in U to

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial u}{\partial x_j}\right) = 0,$$

where $(A_{ij}(x))$ are measurable, with

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{n} A_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2$$

for almost every $x \in U$, some $0 < \lambda, \Lambda < \infty$, and all $\xi \in \mathbf{R}^n$.

(a) Define what is meant by a weak solution/subsolution to the above PDE.

(b) Prove that if u is a bounded weak solution to $Lu = 0$ in U and $\phi \in C^2(-\infty, \infty)$ is convex, then $\phi \circ u$ is a weak subsolution in U. You may assume that $(\phi' \circ u)\zeta \in W_0^{1,2}(U)$ with the usual distributional derivatives for a product whenever $\zeta \in C_0^\infty(U)$.

(8) Let H be a Hilbert space and $B(\cdot,\cdot)$ a bilinear form on H.

(a) State the Lax - Milgram theorem for B, H.

(b) Let $U \subset \mathbf{R}^n$ be a bounded connected open set. Then $u \in W^{1,2}(U)$ is said to be a weak solution to *Neumann's* problem:

$$(+)\begin{cases} -\Delta u = f \text{ in } U \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U \end{cases}$$

provided that

$$\int_U Du \cdot Dv \, dx = \int_U f \, v \, dx$$

for all $v \in W^{1,2}(U)$. Use the Lax - Milgram theorem to show that if $f \in L^2(U)$, then $(+)$ has a weak solution if and only if $\int_U f \, dx = 0$. You may assume that Poincaré's inequality holds for U in the form

$$\int_U |u - u_U|^2 \, dx \leq c \int_U |Du|^2 dx$$

where u_U denotes the average of u on U and c is a positive constant depending only on U.