THE BOUSFIELD-KAN SPECTRAL SEQUENCE

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1. References

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2. Cosimplicial spaces and Tot

2.1. Cosimplicial objects

Recall the finite ordinal category $\mathbf{\Delta}$. Its objects are the finite totally ordered sets $\mathbf{n} = \{0, 1, \ldots, n\}$, and morphisms are the order-preserving maps. For each $0 \leq i \leq n$, there are maps

$$d^i : \mathbf{n-1} \to \mathbf{n} \quad \text{(cofaces)}$$

and

$$s^i : \mathbf{n+1} \to \mathbf{n} \quad \text{(codegeneracies)},$$

where d^i is the injection that omits the element i and s^i is the surjection such that $s^i(i) = s^i(i+1) = i$.

Definition 2.1. Given a category $\mathscr{C}$, a **cosimplicial object** in $\mathscr{C}$ is a functor $\mathbf{\Delta} \to \mathscr{C}$. Cosimplicial objects and natural transformations between them form a category, denoted $c\mathscr{C}$.

Exercise 2.2. If $s\mathscr{C}$ denotes the category of simplicial objects in $\mathscr{C}$, there is a canonical isomorphism $c\mathscr{C} \cong (c(\mathscr{C}^{op}))^{op}$.

Example 2.3. The standard n-simplices $\Delta[n] = \mathrm{Hom}_{\mathbf{\Delta}}(-, \mathbf{n})$, as n varies, constitute a cosimplicial simplicial set.

Example 2.4. The geometric n-simplices $\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1\}$, as n varies, constitute a cosimplicial space.

Example 2.5. Given any $X \in \mathscr{C}$, one can form the *constant*, or *discrete*, cosimplicial object $X^\bullet$ given by $X^n = X$ for all n and by setting all cofaces and codegeneracies to be the identity map of X.

Example 2.6. (Vague) Suppose we have some sort of "simplicial resolution" $X_\bullet \to C$ of some $C \in \mathscr{C}$. Then $\mathrm{Hom}_{\mathscr{C}}(X_\bullet, D)$ forms a cosimplicial set, and if $\mathscr{C}$ is enriched over $\mathscr{D}$, the above yields a cosimplicial object in $\mathscr{D}$.

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Example 2.7. Let $X_\bullet, Y_\bullet \in sSet$. Then we get a cosimplicial simplicial set $\mathrm{Hom}(X_\bullet, Y_\bullet)$ defined by

$$\mathrm{Hom}(X_\bullet, Y_\bullet)^n = \mathrm{Hom}(X_k, Y_\bullet).$$

Proposition 2.8. *Suppose that $\mathscr{C}$ is a closed symmetric monoidal category will all small limits. Then $c\mathscr{C}$ is enriched and bitensored over $\mathscr{C}$; in other words, for each $X^\bullet, Y^\bullet \in c\mathscr{C}$ and $C \in \mathscr{C}$, we have objects $\mathscr{C}(X^\bullet, Y^\bullet)$, $X^\bullet \otimes C$, $(Y^\bullet)^C$ together with natural isomorphisms*

$$\mathrm{Hom}_{\mathscr{C}}(C, \mathscr{C}(X^\bullet, Y^\bullet)) \cong \mathrm{Hom}_{c\mathscr{C}}(X^\bullet \otimes C, Y^\bullet) \cong \mathrm{Hom}_{c\mathscr{C}}(X^\bullet, (Y^\bullet)^C).$$

Proof. We define the objects by

$$(X^\bullet \otimes C)^n = X^n \otimes C,$$

$$\big((Y^\bullet)^C\big)^n = \mathscr{C}(C, Y^n),$$

and

$$\mathscr{C}(X^\bullet, Y^\bullet) = \mathrm{eq}\bigg(\prod_{\mathbf{n}} \mathscr{C}(X^n, Y^n) \rightrightarrows \prod_{\mathbf{m}\to\mathbf{n}} \mathscr{C}(X^m, Y^n)\bigg).$$

We leave the claimed isomorphisms to the reader. ∎

2.2. The homotopy theory of cosimplicial spaces

We will allow "spaces" to mean either topological spaces or simplicial sets, and we will write Spc for the category of spaces.

Recall that Spc is cartesian closed; given $X, Y \in \mathrm{Spc}$, we will as usual write $\mathrm{Map}(X, Y) \in \mathrm{Spc}$ for the internal hom functor. Moreover, Spc has a model structure which is compatible with this monoidal structure. The latter means that given a cofibration $A \xrightarrow{i} B$ and a fibration $X \xrightarrow{p} Y$, the induced map

$$\mathrm{Map}(B, X) \to \mathrm{Map}(A, X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B, Y)$$

is a fibration which is a weak equivalence if either i or p is a weak equivalence. In this section, we will describe how cSpc inherits model structure from Spc. Before we are ready to describe this model structure, we need a few preliminaries.

Definition 2.9. Given $X^\bullet \in c\mathrm{Spc}$, we define

$$\pi^0(X^\bullet) = \mathrm{eq}(X^0 \overset{d^1}{\underset{d^0}{\rightrightarrows}} X^1)$$

in Spc. This is sometimes called the **maximal augmentation** of $X^\bullet$.

Definition 2.10. For $X^\bullet \in c\mathrm{Spc}$ and $n \geq -1$, we define the n**th matching space** $M^n X^\bullet$ by

$$M^n X^\bullet = \lim_{\substack{\mathbf{n+1}\twoheadrightarrow\mathbf{k} \\ \mathbf{k}<\mathbf{n+1}}} X^k \cong \mathrm{eq}\bigg(\prod_{i=0}^{n} X^n \rightrightarrows \prod_{0\leq i<j\leq n} X^{n-1}\bigg),$$

where the two maps are given on the factor $i < j$ by the composites

$$\prod_{i=0}^{n} X^n \xrightarrow{p_j} X^n \xrightarrow{s^i} X^{n-1} \quad \text{and} \quad \prod_{i=0}^{n} X^n \xrightarrow{p_i} X^n \xrightarrow{s^{j-1}} X^{n-1}.$$

Note that by the definition of a cosimplicial object, there is a canonical map $X^{n+1} \to M^n X^\bullet$ given by the codegeneracies $s^i : X^{n+1} \to X^n$.

Remark 2.11. The composite $X^n \xrightarrow{d^i} X^{n+1} \to M^n X$ is given by

$$(d^{i-1}s^0, \ldots, d^{i-1}s^{i-2}, \mathrm{id}, \mathrm{id}, d^i s^i, \ldots, d^i s^{n-1}), \qquad 1 \le i \le n,$$
$$(\mathrm{id}, d^0 s^0, \ldots, d^0 s^{n-1}), \qquad i = 0,$$

and

$$(d^n s^0, \ldots, d^n s^{n-1}, \mathrm{id}), \qquad i = n+1.$$

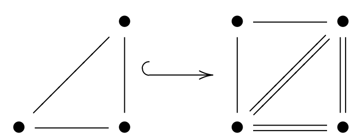
The above formulae allow us to define coface maps $d^i : X^n \to M^n X$ and codegeneracies $s^i : M^n X \to X^n$ without reference to X^{n+1}.

In fact, given an n-truncated cosimplicial object $\{X^k\}_{k \le n}$, the above allows us to form an $n+1$-truncated cosimplicial object $\rho_n X^\bullet$ which is X^k in codegree $k \le n$ and $M^n X$ in codegree $n+1$. Moreover, the functor $\rho_n : c_n\mathscr{C} \to c_{n+1}\mathscr{C}$ from n-truncated cosimplicial objects to $n+1$-truncated cosimplicial objects is right adjoint to the n-truncation functor $\tau_{\le n} : c_{n+1}\mathscr{C} \to c_n\mathscr{C}$, so that we have

$$\mathrm{Hom}_{c_n\mathscr{C}}(\tau_{\le n} X^\bullet, Y^\bullet) \cong \mathrm{Hom}_{c_{n+1}\mathscr{C}}(X^\bullet, \rho_n Y^\bullet).$$

Example 2.12. By definition, $M^{-1}X^\bullet = *$, and $M^0 X^\bullet = X^0$.

Example 2.13. $M^1 X^\bullet = X^1 \times_{X^0} X^1$, where the two maps $X^1 \to X^0$ are s^0 and s^1. Taking $X^\bullet = \Delta^\bullet$, we get $M^1\Delta^\bullet = \Delta^1 \times \Delta^1$. Tracing through the definitions, we see that the canoncial map $\Delta^2 \to M^1\Delta^\bullet$ is



Definition 2.14. We say that a map $f^\bullet : X^\bullet \to Y^\bullet$ in cSpc is a

- **weak equivalence** if each $f^n : X^n \to Y^n$ is a weak equivalence in Spc for $n \ge 0$
- **cofibration** if each $f^n : X^n \to Y^n$ is a cofibration for $n \ge 0$ and f^0 induces an isomorphism $\pi^0(X^\bullet) \cong \pi^0(Y^\bullet)$
- **fibration** if each
$$X^{n+1} \to Y^{n+1} \times_{M^n Y^\bullet} M^n X^\bullet$$
is a fibration in Spc for $n \ge -1$.

Remark 2.15. Taking Spc $= sSet$, we see that $X^\bullet$ is cofibrant if and only if the maximal augmention is empty. In particular, this means that if K is any simplicial set, then the constant cosimplicial space on K is cofibrant if and only if $K = \emptyset$.

On the other hand, the constant cosimplicial space on K is *fibrant* if and only if K is a fibrant space.

Example 2.16. The cosimplicial space $\Delta^\bullet$ is cofibrant, but we see by Example **2.13** above that it is not fibrant.

Theorem 2.17. *With the above definitions,* cSpc *admits the structure of a proper, cofibrantly generated* Spc*-model category.*

2.3. Tot(X)

Definition 2.18. Recall the cosimplicial object $\Delta^\bullet \in c$Spc from above. Given any $X^\bullet \in c$Spc, we define the **total space** $\mathrm{Tot}(X)$ by

$$\mathrm{Tot}(X) = \mathrm{Spc}(\Delta^\bullet, X^\bullet).$$

One should think of the above construction as dual to the realization of a simplicial space.

Example 2.19. Recall from example **2.7** that given $K_\bullet, L_\bullet \in sSet$, we have a cosimplicial simplicial set $\mathrm{Hom}(K_\bullet, L_\bullet)$. We claim that

$$\mathrm{Tot}(\mathrm{Hom}(K_\bullet, L_\bullet)) \cong \mathrm{Map}(K_\bullet, L_\bullet).$$

This follows from the identification

$$K_\bullet \cong \operatorname*{colim}_{\Delta} \Delta^\bullet \times K_\bullet \cong \mathrm{coeq}\Big(\coprod_{\mathbf{m}\to\mathbf{n}} \Delta^m \times K_n \rightrightarrows \coprod_{\mathbf{n}} \Delta^n \times K_n\Big).$$

Indeed, we then have

$$\begin{aligned}
\mathrm{Tot}(\mathrm{Hom}(K_\bullet, L_\bullet)) &= \mathrm{eq}\Big(\prod_{\mathbf{n}} \mathrm{Map}(\Delta^n, \mathrm{Hom}(K_n, L_\bullet)) \rightrightarrows \prod_{\mathbf{m}\to\mathbf{n}} \mathrm{Map}(\Delta^m, \mathrm{Hom}(K_n, L_\bullet))\Big) \\
&\cong \mathrm{eq}\Big(\prod_{\mathbf{n}} \mathrm{Map}(\Delta^n \times K_n, L_\bullet) \rightrightarrows \prod_{\mathbf{m}\to\mathbf{n}} \mathrm{Map}(\Delta^m \times K_n, L_\bullet)\Big) \\
&\cong \mathrm{Map}(K_\bullet, L_\bullet).
\end{aligned}$$

For any $n \geq 0$, we have a cosimplicial space $\mathrm{sk}_n \Delta^\bullet$, given by

$$(\mathrm{sk}_n \Delta^\bullet)^m = \mathrm{sk}_n(\Delta^m)$$

(if $\mathrm{Spc} = Top$, we should probably define $\mathrm{sk}_n(\Delta^m) = |\mathrm{sk}_n \Delta[m]|$). Moreover, we have natural inclusions $\mathrm{sk}_n \Delta^\bullet \hookrightarrow \mathrm{sk}_{n+1} \Delta^\bullet$, which are cofibrations (note that the maximal augmentations are empty).

Definition 2.20. We will write $\mathrm{Tot}_n(X^\bullet) = \mathrm{Spc}(\mathrm{sk}_n \Delta^\bullet, X^\bullet)$.

Remark 2.21. It is not difficult to see that $\mathrm{Tot}_0(X^\bullet) \cong X^0$. This follows from the fact that everything in $\mathrm{sk}_0 \Delta^\bullet$ is in the image of the coface maps, so that maps $\mathrm{sk}_0 \Delta^\bullet \to X^\bullet$ are determined in codegree 0.

The above discussion implies that for each fibrant $X^\bullet$ we have a tower of fibrations

$$\cdots \to \mathrm{Tot}_n(X^\bullet) \to \mathrm{Tot}_{n-1}(X^\bullet) \to \cdots \to \mathrm{Tot}_0(X^\bullet) \cong X^0$$

with $\mathrm{Tot}(X^\bullet) \cong \varprojlim_n \mathrm{Tot}_n(X^\bullet)$.

3. The homotopy spectral sequence of a cosimplicial space

The goal of this section will be to develop the homotopy spectral sequence of a cosimplicial space $X^\bullet$.

3.1. The homotopy spectral sequence for a tower of fibrations

Recall that given a tower of pointed fibrations

$$\cdots \to Y_s \xrightarrow{p_s} Y_{s-1} \xrightarrow{p_{s-1}} Y_{s-2} \to \cdots$$

there is a second octant spectral sequence

$$E_1^{s,t} \cong \pi_{t-s}(F_s) \Rightarrow \pi_{t-s}(Y),$$

where $Y = \varprojlim Y_s$ and $F_s \xrightarrow{i_s} Y_s \xrightarrow{p_s} Y_{s-1}$ is the fiber. This spectral sequence arises from the exact couple

$$\begin{tikzcd}
\cdots \arrow[r, "p_{s*}"] & \pi_*(Y_s) \arrow[r, "p_{s-1*}"] \arrow[dl, dashed, "\partial"'] & \pi_*(Y_{s-1}) \arrow[r, "p_{s-2*}"] \arrow[dl, dashed, "\partial"'] & \cdots \arrow[dl, dashed, "\partial"'] \\
{} & \pi_*(F_s) \arrow[u, "i_{s*}"] & \pi_*(F_{s-1}) \arrow[u, "i_{s-1*}"] &
\end{tikzcd}$$

This exact couple is not an exact couple in abelian groups, since $\pi_1(Z)$ is not always abelian, and $\pi_0(Z)$ is only a pointed set. As a result, we have that in general $E_r^{s,s+1}$ is only a (not necessarily abelian) group and $E_r^{s,s}$ is only a pointed set.

In fact, the usual definition of $E_{r+1}^{s,s+1}$ as $\ker(d_r)/\operatorname{im}(d_r)$ works once one knows that $\operatorname{im}(d_r) < \ker(d_r)$ is normal (in fact central).

For $E_{r+1}^{s,s}$, one first shows that $E_r^{s-r,s-r+1}$ acts on $E_r^{s,s}$ via d_r, but $E_{r+1}^{s,s}$ is *not* defined to be the set of orbits $E_r^{s,s}/d_r(E_r^{s-r,s-r+1})$. Rather, one defines $E_{r+1}^{s,s}$ to be the quotient of "what would have been the cycles under d_r". To be precise, define $Z_r^{s,s} \subseteq E_r^{s,s}$ to be the inverse image under $i_* : \pi_0(F_s) \to \pi_0(Y_s)$ of $\operatorname{im}(\pi_0(Y_{s+r}) \to \pi_0(Y_s)) \subseteq \pi_0(Y_s)$. Then one shows that the action of $E_r^{s-r,s-r+1}$ on $E_r^{s,s}$ preserves $Z_r^{s,s}$, and one defines

$$E_{r+1}^{s,s} = Z_r^{s,s}/d_r(E_r^{s-r,s-r+1}).$$

Due to these, for now, mysterious choices of $E_{r+1}^{s,s}$, we say that the spectral sequence is **fringed** rather than edged. Note that if $\pi_0(Y_{s+1}) \to \pi_0(Y_s)$ is surjective for each s, the fringing disappears. One reason for fringing the spectral sequence is that it makes the following proposition true at the level of π_0.

Proposition 3.1 (BK, IX.4.1)**.** *For each $r \geq 0$ there is a* ***r-th derived homotopy sequence***

$$\cdots \to \pi_{t-s-1}Y_{s-r-1}^{(r)} \to E_{r+1}^{s,t} \to \pi_{t-s}Y_s^{(r)} \to \pi_{t-s}Y_{s-1}^{(r)} \to E_{r+1}^{s+r,t+r-1} \to \pi_{t-s-1}Y_{s+r}^{(r)} \to \ldots,$$

where $\pi_i Y_n^{(r)} = \operatorname{im}(\pi_i Y_{n+r} \to \pi_i Y_n)$.

3.2. Convergence

Now we will address the question of what the above spectral sequence computes.

Recall that for $i \geq 0$ there is a natural short exact sequence

$$* \to \varprojlim{}^1 \pi_{i+1} Y_n \to \pi_i Y \to \varprojlim \pi_i Y_n \to *.$$

Proposition 3.2 (Connectivity Lemma)**.** *Let $k \geq 0$ and $r \geq 1$ such that $E_r^{s,t} = *$ for $k \geq t-s \geq 0$. Then*

$$\varprojlim \pi_i Y_n = * = \varprojlim{}^1 \pi_{i+1} Y_n, \qquad 0 \leq i \leq k.$$

Thus, Y is k-connected.

Definition 3.3. Note that if $r > s$ then $E_r^{s,t}$ cannot be the target of any nontrivial differential, so that $E_{r+1}^{s,t} \subseteq E_r^{s,t}$. We say that $\{E_r^{s,t}\}$ **converges completely** to $\pi_i Y$ $(i \geq 1)$ if

$$\varprojlim{}^1 E_r^{s,s+i} = *, \qquad s \geq 0.$$

Lemma 3.4 (BK, IX.5.4, Complete Convergence Lemma)**.** *If $\{E_r^{s,t}\}$ converges completely to $\pi_i Y$ then*

$$\varprojlim{}^1 \pi_i Y_s = * \qquad \textit{and} \qquad E_r^{s,s+i} \Rightarrow \pi_i Y.$$

3.3. Specializing to the tower of fibrations for $\mathrm{Tot}(X^\bullet)$

Assume that $X^\bullet$ is a fibrant *pointed* cosimplicial space. Then each $\mathrm{Tot}_n(X^\bullet)$ acquires a basepoint, the constant map to the basepoint of $X^\bullet$. In order to apply the above machinery from the previous section to our tower of fibrations

$$\cdots \to \mathrm{Tot}_n(X^\bullet) \to \mathrm{Tot}_{n-1}(X^\bullet) \to \dots,$$

we will first identify the fiber of $\mathrm{Tot}_n(X^\bullet) \to \mathrm{Tot}_{n-1}(X^\bullet)$.

Lemma 3.5. *The fiber F_n of $\mathrm{Tot}_n(X^\bullet) \to \mathrm{Tot}_{n-1}(X^\bullet)$ is $\Omega^n(N^n X^\bullet)$, where*

$$N^n X^\bullet = \ker\left(X^n \xrightarrow{(s^0,\dots,s^{n-1})} \prod_{i=0}^{n-1} X^{n-1} \right) = X^n \cap \ker s^0 \cap \dots \cap \ker s^{n-1}.$$

Sketch. We are interested in the space $F_n = \mathrm{Spc}(\mathrm{sk}_n \Delta^\bullet / \mathrm{sk}_{n-1} \Delta^\bullet, X^\bullet)$. Now the cosimplicial space $\mathrm{sk}_n \Delta^\bullet / \mathrm{sk}_{n-1} \Delta^\bullet$ is a point in codegree less than n and the n-sphere $S^n = \Delta^n/\partial\Delta^n$ in codegree n. Moreover, everything in codegree bigger than n is in the image of a coface map since this is true for $\mathrm{sk}_n \Delta^\bullet$ (for instance, $\mathrm{sk}_n \Delta^{n+1} = \partial\Delta^{n+1}$ is the union of the cofaces $d^i(\Delta^n)$). It follows that maps of cosimplicial spaces

$$f : \mathrm{sk}_n \Delta^\bullet / \mathrm{sk}_{n-1} \Delta^\bullet \to X^\bullet$$

correspond to maps of spaces $f^n : S^n \to X^n$ such that $s^i \circ f^n = *$ for all i. ■

The homotopy spectral sequence of the pointed cosimplical space $X^\bullet$ is the spectral sequence for the tower of fibrations $\mathrm{Tot}_*(X^\bullet)$. The lemma allows us to identify

$$E_1^{s,t} = \pi_{t-s} F_s \cong \pi_t N^s X^\bullet.$$

The reason for the choice of notation $N^s X^\bullet$ will be made apparent in a moment.

For a cosimplicial pointed set $Z^\bullet$, we define $N^s Z^\bullet$ as above by the formula

$$N^s Z^\bullet = \bigcap_{i=0}^{s-1} \ker\bigl(s^i : Z^s \to Z^{s-1}\bigr).$$

Note that if $G^\bullet$ is a cosimplicial *abelian group*, then the cochain complex $NG^\bullet$, with differential given by $\sum(-1)^i d^i$, is the analogue of the normalized chain complex associated to a simplicial abelian group. Moreover, as in the simplicial case we have

$$\mathrm{H}^n(N^*G, \sum(-1)^i d^i) \cong \mathrm{H}^n(G^*, \sum(-1)^i d^i).$$

Lemma 3.6. *We have isomorphisms*

$$\pi_t N^s X^\bullet \cong N^s \pi_t X^\bullet, \qquad t \geq s \geq 0.$$

Definition 3.7. (1) For a cosimplicial abelian group $A^\bullet$, we define **cohomotopy groups** $\pi^s A$ by

$$\pi^s A = \mathrm{H}^s\bigl(A, \sum(-1)^i d^i\bigr), \qquad s \geq 0.$$

(2) In the case of a not necessarily abelian cosimplicial group, we can nevertheless still define π^0 and π^1. The group $\pi^0 G^\bullet$ is defined as

$$\pi^0(G^\bullet) = \mathrm{eq}(G^0 \underset{d^0}{\overset{d^1}{\rightrightarrows}} G^1).$$

To define the pointed set $\pi^1 G^\bullet$, first set $Z^1 G^\bullet = \{g \in G^1 \mid d^2(g)(d^1(g))^{-1} d^0(g) = e\}$. We define an action of G^0 on $Z^1 G^\bullet$ by $g \cdot z = d^0(g) z d^1(g)^{-1}$. Finally, set

$$\pi^1 G^\bullet = Z^1 G^\bullet / G^0.$$

(3) For a cosimplicial set $X^\bullet$, as before we define

$$\pi^0(X^\bullet) = \mathrm{eq}(G^0 \underset{d^0}{\overset{d^1}{\rightrightarrows}} X^1).$$

Proposition 3.8 (GJ, VIII.1.15). *The differential* $d_1 : N^s\pi_t X^\bullet \to N^{s+1}\pi_t X^\bullet$ *is* $\sum(-1)^i d^i$, *and we have*

$$E_2^{s,t} \cong \pi^s N^\bullet \pi_t X^\bullet \cong \pi^s\pi_t X^\bullet.$$

Note that if the spectral sequence were not fringed, then we would not have the nice formula $E_2^{s,s} = \pi^s\pi_s X^\bullet$.

3.4. An example

Returning to Example **2.19**, recall that if $K_\bullet, L_\bullet \in sSet$ we have a cosimplicial simplicial set $\mathrm{Hom}(K_\bullet, L_\bullet)$ and that

$$\mathrm{Tot}(\mathrm{Hom}(K_\bullet, L_\bullet)) \cong \mathrm{Map}(K_\bullet, L_\bullet).$$

Moreover, the same argument that gave the above identification yields an identification

$$\mathrm{Tot}_n(\mathrm{Hom}(K_\bullet, L_\bullet)) = \mathrm{Spc}(\mathrm{sk}_n\,\Delta^n, \mathrm{Hom}(K_\bullet, L_\bullet)) \cong \mathrm{Map}(\mathrm{sk}_n\,K_\bullet, L_\bullet).$$

There is also the pointed variant in which one begins with $K_\bullet, L_\bullet \in sSet_*$ and forms the pointed cosimplicial simplicial set of pointed maps $\mathrm{Hom}_*(K_\bullet, L_\bullet)$; one then has as above

$$\mathrm{Tot}(\mathrm{Hom}_*(K_\bullet, L_\bullet)) \cong \mathrm{Map}_*(K_\bullet, L_\bullet)$$

and

$$\mathrm{Tot}_n(\mathrm{Hom}_*(K_\bullet, L_\bullet)) \cong \mathrm{Map}_*(\mathrm{sk}_n\,K_\bullet, L_\bullet).$$

We will write $X^n = \mathrm{Hom}_*(K_n, L_\bullet)$. Then

$$X^n = \mathrm{Hom}_*(K_n, L_\bullet) = (L_\bullet)^{\tilde{K}_n},$$

where $\tilde{K}_n = K_n - *$.

In order for $X^\bullet$ to be fibrant, one must assume that $L_\bullet$ is a pointed Kan complex. Now the above spectral sequence has

$$E_1^{s,t} = N^s\pi_t X^\bullet = N^s\pi_t(L_\bullet^{\tilde{K}_\bullet}) \cong N^s(\pi_t(L_\bullet))^{\tilde{K}_\bullet},$$

and Proposition **3.8** allows us to deduce that

$$E_2^{s,t} \cong \tilde{\mathrm{H}}^s(K_\bullet; \pi_t(L_\bullet)).$$

Assuming, for example, that $K_\bullet$ is finite-dimensional, so that $\mathrm{sk}_n\,K_\bullet \cong K_\bullet$ for some n, we get that the spectral sequence collapses and $\tilde{\mathrm{H}}^s(K_\bullet, \pi_t(L_\bullet)) \Rightarrow \pi_{t-s}\,\mathrm{Map}_*(K_\bullet, L_\bullet)$.

4. Obstruction Theory

Recall that in the previous section, we discussed the homotopy spectral sequence for the tower of fibrations

$$\cdots \to \mathrm{Tot}\,n+1(X^\bullet) \to \mathrm{Tot}_n(X^\bullet) \to \dots$$

for a *pointed* (fibrant) cosimplicial space $X^\bullet$. The point is that the basepoint of $X^\bullet$ gives basepoints to each $\mathrm{Tot}_n(X^\bullet)$ and to $\mathrm{Tot}(X^\bullet)$ such that all of the maps $\mathrm{Tot}_n(X^\bullet) \to \mathrm{Tot}_{n-1}(X^\bullet)$ preserve these basepoints. In this section, we will *not* assume given a basepoint for $X^\bullet$. Note that the discussion of the previous sections does not a priori apply; even if one chooses basepoints for each X^n, there is no reason for these choices to be compatible with respect to the various cosimplicial structure maps. Following Bousfield, we will develop some obstruction theory to deal with the problem of succussively choosing basepoints in $\mathrm{Tot}_n(X^\bullet)$ that lift to basepoints in $\mathrm{Tot}_{n+1}(X^\bullet)$.

4.1. $n = 0$

Let us start at the bottom, where everthing is simplest. Thus assume given some basepoint $x \in \mathrm{Tot}_0(X^\bullet) \cong X^0$. We want to know if this point is in the image of $\mathrm{Tot}_1(X^\bullet) \to \mathrm{Tot}_0(\bullet)$.

The first step will be to get a more concrete understanding of the space $\mathrm{Tot}_1(X^\bullet)$. Using the explicit formula for $\mathrm{Tot}_1(X^\bullet)$ from Proposition **2.8** and an argument as in Remark **2.21**, we get the formula

$$\mathrm{Tot}_1(X^\bullet) = \mathrm{eq}\left(\prod_{\mathbf{i}=0}^{1} \mathrm{Map}(\Delta^i, X^i) \rightrightarrows \prod_{0\le \mathbf{i}\to\mathbf{j}\le 1} \mathrm{Map}(\Delta^i, X^j)\right).$$

Explicitly, a point in $\mathrm{Tot}_1(X^\bullet)$ consists of a point $x \in X^0$ and a path γ in X^1 such that

- $\gamma(0) = d^0(x)$,
- $\gamma(1) = d^1(x)$, and
- $s^0 \circ \gamma$ is the constant loop at x.

Then the map $\mathrm{Tot}_1(X^\bullet) \to \mathrm{Tot}_0(X^\bullet) = X^0$ is given by $(x, \gamma) \mapsto x$. Moreover, we see that this map fits into a pullback square

$$\begin{tikzcd}
\mathrm{Tot}_1(X^\bullet) \arrow[r] \arrow[d] & \mathrm{Map}(\Delta^1, X^1) \arrow[d] \\
\mathrm{Tot}_0(X^\bullet) \arrow[r] & \mathrm{Map}(\partial\Delta^1, X^1) \times_{\mathrm{Map}(\partial\Delta^1, X^0)} \mathrm{Map}(\Delta^1, X^0),
\end{tikzcd}$$

where the bottom horizontal map is given by

$$x \mapsto ((d^0(x), d^1(x)), (\text{constant loop at } x)$$

and the right vertical map is

$$\gamma \mapsto (\gamma_{|\partial\Delta^1}, s^0 \circ \gamma).$$

Thus the question of the existence of a lift of $x \in \mathrm{Tot}_0(X^\bullet)$ to $\mathrm{Tot}_1(X^\bullet)$, is equivalent to the existence of a path γ in X^1 such that $\gamma(0) = d^0(x)$, $\gamma(1) = d^1(x)$, and $s^0 \circ \gamma$ is constant at x.

In fact, it turns out that the last condition is not needed. Suppose given a path γ in X^1 such that $\gamma(0) = d^0(x)$ and $\gamma(1) = d^1(x)$. Then $s^0\gamma$ is a loop in X^0 at x, and in general there is no reason for it to be constant.

Letting Λ^2_0 be the $(2,0)$-horn (the simplicial set Δ^2 without the 2-simplex i_2 and the face $d_0(i_2)$), define a map $\Lambda^2_0 \to X^1$ by using the path γ on the face $d_1(i_2)$ and $d^0(s^0\gamma)$ on the face $d_2(i_2)$:

$$\begin{tikzcd}
 & d^1(x) \\
d^0(x) \arrow[ur, "\gamma"] \arrow[r, "d^0(s^0\gamma)"'] & d^0(x)
\end{tikzcd}$$

Define a map $\Delta^2 \to X^0$ to be $\Delta^2 \xrightarrow{s^0} \Delta^1 \xrightarrow{s^0\gamma} X^0$. Then one can check that the diagram

$$\begin{tikzcd}
\Lambda^2_0 \arrow[r] \arrow[d] & X^1 \arrow[d] \\
\Delta^2 \arrow[r] \arrow[ur, dashed] & X^0
\end{tikzcd}$$

commutes. Moreover, we get a lifting $\sigma : \Delta^2 \to X^1$ since $\Lambda^2_0 \hookrightarrow \Delta^2$ is an acyclic cofibration and $X^1 \to X^0$ is a fibration (we are assuming that $X^\bullet$ is a fibrant cosimplicial space which means, in particular, that $X^1 \to X^0$ is a fibration). Finally, define $\tilde{\gamma} = \sigma_{|d_0(i_2)}$

$$\begin{tikzcd} & d^1(x) \\ d^0(x) \arrow[ur, "\gamma"] \arrow[r, "d^0(s^0\gamma)"'] & d^0(x). \arrow[u, "\tilde{\gamma}"'] \end{tikzcd}$$

By construction, $\tilde{\gamma}$ is a path from $d^0(x)$ to $d^1(x)$ such that $s^0\tilde{\gamma}$ is constant at x. We have just proved

Proposition 4.1 (Lifting criterion)**.** *Let $x \in \mathrm{Tot}_0(X^\bullet) \cong X^0$. Then x lifts to $\mathrm{Tot}_1(X^\bullet)$ if and only if there is a path from $d^0(x)$ to $d^1(x)$, in other words, if and only if $[d^0(x)] = [d^1(x)]$ in $\pi_0(X^1)$.*

Note that another way to say that $[d^0(x)] = [d^1(x)]$ in $\pi_0(X^1)$ is that $[x] \in \pi^0\pi_0(X^\bullet)$. Although it does not make sense general, we will say that

> "the element $[d^0(x)][d^1(x)]^{-1} \in \pi_0(X^1)$ is the obstruction to the lifting of x to $\mathrm{Tot}_1(X^\bullet)$".

Moreover, thinking back to the spectral sequence from the previous section, this mythical obstruction element lives in $E_1^{1,0}$.

4.2. $n > 0$

As it turns out, the results from the previous section generalize.

For instance, one can see that a point of $\mathrm{Tot}_n(X^\bullet)$ consists of $(x_0, \ldots, x_n)$, with $x_k : \Delta^k \to X^k$ such that

- $d^i(x_k) = x_{k+1} \circ d^i$ for $0 \le i \le k+1$ and $0 \le k \le n-1$ and
- $s^j(x_k) = x_{k-1} \circ s^j$ for $0 \le j \le k-1$ and $1 \le k \le n$.

The map $\mathrm{Tot}_n(X^\bullet) \to \mathrm{Tot}_{n-1}(X^\bullet)$ is simply forgetting x_n, and one has a pullback square

$$\begin{tikzcd} \mathrm{Tot}_n(X^\bullet) \arrow[r] \arrow[d] & \mathrm{Map}(\Delta^n, X^n) \arrow[d] \\ \mathrm{Tot}_{n-1}(X^\bullet) \arrow[r] & \mathrm{Map}(\partial\Delta^n, X^n) \times_{\mathrm{Map}(\partial\Delta^n, M^{n-1}X^\bullet)} \mathrm{Map}(\Delta^n, M^{n-1}X^\bullet). \end{tikzcd}$$

The result here is

Proposition 4.2 (Lifting criterion)**.** *Let $x \in \mathrm{Tot}_{n-1}(X^\bullet)$. Then x lifts to $\mathrm{Tot}_n(X^\bullet)$ if and only if the induced map $o(x) : \partial\Delta^n \to X^n$ represents the trivial element in $\pi_{n-1}(X^n, d^0 \ldots d^0(x_0))$.*

We call $o(x)$ the obstruction to lifting x. As in the previous section, we think of this element as living in $E_1^{n+1,n}$.

4.3. Uniqueness

Now suppose that $x \in \mathrm{Tot}_{n-1}(X^\bullet)$ has a lift $y \in \mathrm{Tot}_n(X^\bullet)$. One can then repeat the above proceedure to determine whether y lifts to $\mathrm{Tot}_{n+1}(X^\bullet)$. However, the obstruction $o(y)$ to lifting y may very well depend on the choice of lift y. That is, bad choices of y might prevent us from lifting x to $\mathrm{Tot}_{n+2}(X^\bullet)$.

Fortunately, there is a nice answer to this problem. First, one shows that the element $o(y) \in \pi_n(X^{n+1}, x_0)$ is a cocycle, so that one may consider the corresponding class

$$[o(y)] \in \pi^{n+1}\pi_n(X^\bullet, x_0).$$

Next, one shows that this class is *independent of y!*. Finally, Bousfield's result is

Theorem 4.3. *Suppose that $x \in \mathrm{Tot}_{n-1}(X^\bullet)$ lifts to $y \in \mathrm{Tot}_n(X^\bullet)$. Then*

(1) *if $n = 1$ and the fundamental groupoid πX^m acts trivially on $\pi_1(X^m, z)$ for all $m \geq 0$, then the element $[o(y)] \in \pi^2\pi_1(X^\bullet, x_0)$ is trivial if and only if x extends to $\mathrm{Tot}_2(X^\bullet)$*

(2) *if $n > 1$ then the element $[o(y)] \in \pi^{n+1}\pi_n(X^\bullet, x_0)$ is trivial if and only if x extends to $\mathrm{Tot}_{n+1}(X^\bullet)$.*

Corollary 4.4. *Let $x \in \mathrm{Tot}_0(X^\bullet) \cong X^0$ lie in $\pi^0\pi_0(X^\bullet)$. If the fundamental groupoid πX^m acts trivially on $\pi_1(X^m, z)$ for all $m \geq 0$ and $\pi^{n+1}\pi_n(X^\bullet, x) = 0$ for $n \geq 1$, then x lifts to $\mathrm{Tot}(X^\bullet)$.*

4.4. Application

We will *very* briefly sketch an application; some of the ideas introduced will be the subject of Niles' talk next time.

Fix a prime p. Let $\mathbb{F}_p : sSet \to sSet$ be the free simplicial $\mathbb{F}_p$-vector space functor. Let Y be a space and let $\mathbb{F}_p^\bullet Y$ be the Bousfield-Kan resolution of Y given by $(\mathbb{F}_p^\bullet Y)^n = \mathbb{F}_p^{n+1} Y$. The functor $\mathbb{F}_p$ is a monad and so yields an augmented cosimplicial space with cofaces given by

$$d^i = \mathbb{F}_p^i \eta \mathbb{F}_p^{n+1-i} : (\mathbb{F}_p Y)^n \to (\mathbb{F}_p Y)^{n+1}$$

and codegeneracies by

$$s^i = \mathbb{F}_p^i \epsilon \mathbb{F}_p^{n-i} : (\mathbb{F}_p Y)^{n+1} \to (\mathbb{F}_p Y)^n.$$

Bousfield and Kan define the p-completion of Y to be $Y_p = \mathrm{Tot}(F_p^\bullet Y)$.

Given another space X, we can form the cosimplical space $\mathrm{Map}(X, \mathbb{F}_p^\bullet Y)$, where the cosimplicial structure comes from $\mathbb{F}_p^\bullet$. We then have

$$\mathrm{Tot}(\mathrm{Map}(X, \mathbb{F}_p^\bullet Y)) \cong \mathrm{Map}(X, \mathrm{Tot}(\mathbb{F}_p^\bullet Y)) \cong \mathrm{Map}(X, Y_p).$$

For any space Z, we have a map

$$\pi_0(\mathrm{Map}(X, Z)) \to \mathrm{Hom}_{\mathcal{K}}(H^* Z, H^* X),$$

where $\mathcal{K}$ is the category of unstable algebras over the mod p Steenrod algebra. This map is an isomorphism if Z is a simplicial $\mathbb{F}_p$-vector space and both X and Z have finite-dimensional H^n for all n. In fact, choosing a basepoint $f \in \mathrm{Map}(X, Z)$ yields a similar isomorphism

$$\pi_t(\mathrm{Map}(X, Z), f) \to \mathrm{Hom}_{\mathcal{K}}(H^* Z, H^*(S^t) \otimes H^* X)$$

under the same assumptions.

Proposition 4.5. *Let X and Y be spaces of finite type (meaning that H^n is finite dimensional for each n). Then a morphism $\varphi : H^* Y \to H^* X$ in $\mathcal{K}$ can be lifted to a map of spaces $X \to Y_p$ if*

$$R^{s+1}\,\mathrm{Der}_{\mathcal{K}}(H^* Y, \Sigma^s H^* X; \varphi) = \pi^{s+1}\,\mathrm{Hom}_{\mathcal{K}}(H^* \mathbb{F}_p^\bullet Y, H^*(S^s \times X)) \cong 0$$

for $s \geq 1$.