THE CALCULUS OF FUNCTORS

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1. The Calculus of Functions

Let $f : \mathbb{R} \to \mathbb{R}$ be a C^∞ function. Given any $a \in \mathbb{R}$ (e.g. $a = 0$), we can approximate the function f by its Taylor series

$$f(x) \sim f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots.$$

For x "close enough" to a, the Taylor series will converge to $f(x)$, provided that $f(x)$ is "analytic". Letting $P_n(f)(x)$ denote the nth Taylor polynomial of $f(x)$, another way to say this is that

$$f(x) = \lim_n P_n(f)(x).$$

2. The Calculus of Functors

Let $F : \mathscr{C} \longrightarrow \mathscr{D}$ be a homotopy functor between pointed model categories.

Notation. We will denote by $\mathbf{Top}_*$ the category of based spaces and more generally by $\mathbf{Top}_Y$ the category of spaces over and under a given space Y. The category of spectra will be denoted $\mathbf{Sp}$, and $\mathbb{S}$ will denote the sphere spectrum.

A ring, or simplicial ring, will always mean either a brave new ring or a simplicial ring or a dg-ring. If R is a commutative ring in any of these senses, we will write $\mathbf{Alg}_R$ and $\mathbf{ComAlg}_R$ for the categories of *augmented* R-algebras and commutative R-algebras, respectively. $\mathbf{Mod}_R$ will denote the category of (left) R-modules.

Example 1. Letting $\mathscr{C}$ or $\mathscr{D}$ be $\mathbf{Top}_*$ or $\mathbf{Sp}$, we have functors

$$\mathrm{Id}_{\mathbf{Top}_*} : \mathbf{Top}_* \longrightarrow \mathbf{Top}_*,$$
$$\Sigma^\infty : \mathbf{Top}_* \longrightarrow \mathbf{Sp},$$
$$\Omega^\infty : \mathbf{Sp} \longrightarrow \mathbf{Top}_*,$$

and

$$\mathrm{Id}_{\mathbf{Sp}} : \mathbf{Sp} \longrightarrow \mathbf{Sp}.$$

It will turn out that all of these functors are "analytic" and that in fact, apart from $\mathrm{Id}_{\mathbf{Top}_*}$, they are all "degree 1".

Example 2. For the algebraic analogue of the previous example, we use the fact that the stabilization of $\mathbf{ComAlg}_R$ is $\mathbf{Mod}_R$ (see [BM]). We have functors

$$\mathrm{Id}_{\mathbf{ComAlg}} : \mathbf{ComAlg}_R \longrightarrow \mathbf{ComAlg}_R,$$
$$\mathbf{TAQ} : \mathbf{ComAlg}_R \longrightarrow \mathbf{Mod}_R,$$
$$\mathscr{Z} : \mathbf{Mod}_R \longrightarrow \mathbf{ComAlg}_R,$$
$$\mathrm{Id}_{\mathbf{Mod}} : \mathbf{Mod}_R \longrightarrow \mathbf{Mod}_R,$$

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where **TAQ** is Topological André-Quillen Homology and $\mathcal{Z}(M)$ is the square-zero extension $R \oplus M$ of R.

Example 3. One of the first examples studied by Goodwillie is Waldhausen's **A**-theory functor from (based) spaces to spectra, which can be defined as

$$\mathbf{A}(X) = \mathbb{K}(\Sigma^\infty(\Omega X)_+).$$

In particular,

$$\mathbf{A}(*) = \mathbb{K}(\Sigma^\infty(*_+)) = \mathbb{K}(\mathbb{S}).$$

3. Degree n functors

We will define what it means for a functor to be "degree (at most) n". Well, actually today we only have time for $n = 0$ and $n = 1$. For any functor F, there is a universal degree n approximation $F \longrightarrow P_n(F)$. Furthermore, the degree n approximation of $P_{n+1}(F)$ is again $P_n(F)$, and there results a tower of polynomial approximations to F:

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & P_3(F) \arrow[d] \\
 & P_2(F) \arrow[d] \\
F \arrow[ruu] \arrow[ru] \arrow[r] \arrow[rd] & P_1(F) \arrow[d] \\
 & P_0(F).
\end{tikzcd}$$

The analogue of the Taylor *series* is the homotopy limit of this tower

$$P_\infty(F) = \operatorname*{holim}_n P_n(F).$$

3.1. Degree 0

Let's start with the notion of a degree 0 functor. This is easy: a degree 0 polynomial function is just a constant function, and we similarly define a functor to be degree 0 if it is constant.

While we're at it, we might as well describe the degree 0 approximation to a functor F: simply evaluate F at the basepoint in $\mathscr{C}$: $P_0(F)(X) = F(*)$. Since $*$ is terminal in $\mathscr{C}$, any X will map to $*$, and applying F to this map provides the desired map

$$F(X) \longrightarrow F(*) =: P_0(F)(X).$$

A functor is said to be **reduced** if $P_0(F) = *$; that is, if F takes value $*_{\mathscr{D}}$ at $*_{\mathscr{C}}$. Note that the fiber of $F \longrightarrow P_0(F)$ defines a reduced functor.

3.2. Degree 1

Before discussing degree 1 functors in general, we will first discuss degree 1 functors that are reduced (these are usually called **linear** or **homogeneous of degree** 1). A linear function f is characterized by the fact that

$$f(x+y) = f(x) + f(y),$$

so one might ask for a functor that behaves similarly. But now one has a choice of how to interpret '+', and in fact the different choices lead to different flavors of Calculus. One could interpret the sum on both sides of the equation as a coproduct or as a product. One could also choose to interpret one as a coproduct and the other as a product.

The usual definition (dropping the assumption that the functor is reduced) is as follows:

Definition. A homotopy functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is said to be **degree** 1, or 1-**excisive**, if whenever

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is a homotopy pushout diagram in $\mathscr{C}$, then

$$\begin{tikzcd} F(A) \arrow[r] \arrow[d] & F(B) \arrow[d] \\ F(C) \arrow[r] & F(D) \end{tikzcd}$$

is a homotopy pullback diagram in $\mathscr{D}$. More succinctly, F is degree 1 if it takes (homotopy) pushouts to (homotopy) pullbacks.

A particular case of a homotopy pushout diagram is, for any $X \in \mathscr{C}$, the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$

Any reduced degree 1 functor must then produce a homotopy pullback diagram

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & F(\Sigma X) \end{tikzcd}$$

In other words, we find that

$$F(X) \simeq \Omega F(\Sigma X).$$

For a general F, we have only a map

$$\begin{tikzcd} F(X) \arrow[r] & \text{holim} \arrow[r] \arrow[d] & F(*) \arrow[d] \\ & F(*) \arrow[r] & F(\Sigma X) \end{tikzcd}$$

To produce a degree 1 approximation to F, we want to force these maps to become equivalences, and so we simply use the standard trick of passing to (homotopy) colimits over these maps.

Letting $T_1(F)(X)$ denote the homotopy limit in the above diagram, we define the degree 1 approximation $P_1(F)$ by the formula

$$P_1(F)(X) = \operatorname*{hocolim}_n (F(X) \to T_1(F)(X) \to T_1(T_1(F))(X) \to \dots).$$

The associated reduced degree 1 functor, denoted $D_1(F)(X)$ sits in a fiber sequence

$$D_1(F)(X) \longrightarrow P_1(F)(X) \longrightarrow P_0(F)(X).$$

In terms of Taylor polynomials of functions, $P_1(F)(X)$ plays the role of $f(a) + f'(a)(x-a)$ and $D_1(F)(X)$ plays the role of $f'(a)(x-a)$. In light of this, it is reasonable to ask what plays the role of the Taylor coefficient $f'(a)$.

Example 4. Since **Sp** is *stable*, homotopy pushout squares coincide with homotopy pullback squares, so $\mathrm{Id}_{\mathbf{Sp}}$ is a linear functor. Similarly, one can see that Σ^∞ and Ω^∞ are linear functors.

However, $\mathrm{Id}_{\mathbf{Top}_*}$ is certainly *not* a linear functor, as homotopy pushout squares in spaces are rarely homotopy pullback squares. According to the definition, $T_1(\mathrm{Id}_{\mathbf{Top}_*})(X) = \Omega\Sigma X$, so that

$$P_1(\mathrm{Id}_{\mathbf{Top}_*})(X) = \operatorname*{hocolim}_n \Omega^n \Sigma^n X = \Omega^\infty \Sigma^\infty(X).$$

Example 5. The answer in the algebraic example is similar. Since $\mathbf{Mod}_R$ is stable, all three functors involving this category are linear. The identity functor on $\mathbf{ComAlg}_R$ is not linear, and it turns out that

$$P_1(\mathrm{Id}_{\mathbf{ComAlg}_R})(B) = \mathcal{Z} \circ \mathbf{TAQ}(B).$$