THE CALCULUS OF FUNCTORS
TALK 3

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1. Degree n Functors

Last time, Rosona told us about degree n functors. The definition was as follows: a homotopy functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is **n-excisive**, or **degree** n, if for every object X_0 and $(n+1)$-tuple $(X_0 \to X_i)_{1 \le i \le n+1}$ of cofibrations with domain X_0, then the homotopy pushout $(n+1)$-cube

$$\begin{tikzcd}
X_0 \arrow[rrr] \arrow[dr] \arrow[dd] & & & X_1 \arrow[dr] \arrow[dd, dashed] & \\
& \cdots \arrow[rrr] \arrow[dd] & & & X_1 \cup_{X_0} X_2 \arrow[dd] \\
X_{n+1} \arrow[rrr, dashed] \arrow[dr] & & & X_1 \cup_{X_0} X_{n+1} \arrow[dr, dashed] & \\
& X_{n+1} \cup_{X_0} X_n \arrow[rrr] & & & \bigcup\limits_{1 \le i \le n+1} X_i
\end{tikzcd}$$

is taken to a homotopy cartesian $(n+1)$-cube

$$\begin{tikzcd}
F(X_0) \arrow[rrr] \arrow[dr] \arrow[dd] & & & X_1 \arrow[dr] \arrow[dd, dashed] & \\
& \cdots \arrow[rrr] \arrow[dd] & & & F(X_1 \cup_{X_0} X_2) \arrow[dd] \\
F(X_{n+1}) \arrow[rrr, dashed] \arrow[dr] & & & F(X_1 \cup_{X_0} X_{n+1}) \arrow[dr, dashed] & \\
& F(X_{n+1} \cup_{X_0} X_n) \arrow[rrr] & & & F\left(\bigcup\limits_{1 \le i \le n+1} X_i\right).
\end{tikzcd}$$

By the way, $(n+1)$-cubes weakly equivalent to the first kind above are called **strongly homotopy cocartesian** $(n+1)$-cubes. Note that for a 2-cube, this is the same as being homotopy cartesian. Any 1-cube (namely, a map) is of this form and so counts as strongly homotopy cocartesian.

Any degree n polynomial can be thought of as a degree $n+1$ polynomial, and the same is true of functors:

Proposition 1. *Let $F : \mathscr{C} \longrightarrow \mathscr{D}$ be degree n. Then F is also degree $n+1$.*

Date: June 27, 2010.

Proof. We will show that every degree 1 functor is degree 2. Suppose given a strongly homotopy cocartesian 3-cube

$$
\begin{tikzcd}
X_0 \arrow[rr] \arrow[dr] \arrow[dd] & & X_1 \arrow[dr] \arrow[dd, dashed] & \\
& X_2 \arrow[rr] \arrow[dd] & & X_1 \cup_{X_0} X_2 \arrow[dd] \\
X_3 \arrow[rr, dashed] \arrow[dr] & & X_1 \cup_{X_0} X_3 \arrow[dr, dashed] & \\
& X_2 \cup_{X_0} X_3 \arrow[rr] & & X_1 \cup_{X_0} X_2 \cup_{X_0} X_3.
\end{tikzcd}
$$

Then, since each face of this 3-cube is a homotopy pushout square, it follows that each face of

$$
\begin{tikzcd}
F(X_0) \arrow[rr] \arrow[dr] \arrow[dd] & & F(X_1) \arrow[dr] \arrow[dd, dashed] & \\
& F(X_2) \arrow[rr] \arrow[dd] & & F(X_1 \cup_{X_0} X_2) \arrow[dd] \\
F(X_3) \arrow[rr, dashed] \arrow[dr] & & F(X_1 \cup_{X_0} X_3) \arrow[dr, dashed] & \\
& F(X_2 \cup_{X_0} X_3) \arrow[rr] & & F(X_1 \cup_{X_0} X_2 \cup_{X_0} X_3)
\end{tikzcd}
$$

is a homotopy pullback square (since F is assumed to be degree 1). This implies, in turn, that the 3-cube is homotopy cartesian.

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2. The Degree n-approximation

We've already seen that for any homotopy functor $F : \mathscr{C} \to \mathscr{D}$ there are functors $P_0(F)$ and $P_1(F)$, the degree 0 and degree 1 approximations to F. These come with natural maps

$$
\begin{tikzcd}
F \arrow[r, "p_0 F"] \arrow[dr, "p_1 F"] & P_1(F) \arrow[d] \\
& P_0(F)
\end{tikzcd}
$$

which are universal maps from F into polynomial functors of the corresponding degrees. The definition of $P_0(F)$ was simply $P_0(F)(X) = F(*)$, the value of F at the basepoint. The degree 1 approximation $P_1(F)$ was constructed as a certain homotopy colimit.

The construction of $P_n(F)$ is similarly as a certain homotopy colimit. Given $X \in \mathscr{C}$, consider the strongly homotopy cocartesian $(n+1)$-cube formed using the $n+1$ copies of the map $X \to C(X)$. Define an object $T_n(F)(X)$ to be the homotopy limit of the diagram obtained by applying F to this $(n+1)$-cube and removing the initial vertex $F(X)$. By definition, there is a canonical map

$$t_n F : F(X) \longrightarrow T_n(F)(X).$$

Applying this construction to the new functor $T_n(F)$ gives a map

$$t_n T_n F : T_n(F)(X) \longrightarrow T_n(T_n(F))(X).$$

The degree n approximation $P_n(F)$ is then defined as the homotopy colimit

$$P_n(F)(X) := \operatorname{hocolim}\big(F(X) \xrightarrow{(t_n F)(X)} T_n(F)(X) \xrightarrow{(t_n T_n F)(X)} T_n(T_n(F))(X) \longrightarrow \dots\big).$$

Example 1. When $n = 1$, this agrees with the previous definition. $T_1(F)(X)$ is the homotopy limit of

$$\begin{tikzcd} & F(CX) \arrow[d] \\ F(CX) \arrow[r] & F(\Sigma X). \end{tikzcd}$$

Theorem 1 (Goodwillie, Calc III, Theorem 1.8)**.** *For any functor F, the functor $P_n(F)$ is degree n.*

The key step in the proof of this theorem is the following lemma.

Lemma 1 (Goodwillie, Calc III, Lemma 1.9)**.** *Let $\mathcal{X}$ be any strongly cocartesian $(n+1)$-cube. Then the map of cubes*

$$F(\mathcal{X}) \xrightarrow{t_nF} (T_nF)(\mathcal{X})$$

factors through some cartesian $(n+1)$-cube.

By his own admittance, Goodwillie's proof of this lemma is "a little opaque". A streamlined proof was later given by Charles Rezk. The cartesian cube is simply the cube $F(\mathcal{X})$ with the initial vertex $F(X_0)$ replaced by the homotopy limit, and the work is in showing that T_nF factors throught this particular cartesian cube.

Given the lemma, it then follows easily that $P_n(F)$ is n-excisive. Given any strongly cocartesian cube $\mathcal{X}$, the cube $P_n(F)(\mathcal{X})$ is a sequential hocolim of cubes $T_n(F)(\mathcal{X})$. By the lemma, this hocolim can be replaced by a sequential hocolim of *cartesian* cubes and is thus cartesian.

The universality of $P_n(F)$ is argued as follows. Note that if the functor F is already degree n, then the map

$$t_nF : F(X) \longrightarrow (T_nF)(X)$$

is an equivalence. This implies that the map

$$p_nF : F(X) \longrightarrow (P_nF)(X)$$

is also an equivalence for F of degree n.

Now suppose given a map $\alpha : F \longrightarrow G$, where G is degree n. Then we can consider the diagram of functors

$$\begin{tikzcd} F \arrow[r, "\alpha"] \arrow[d, "p_nF"'] & G \arrow[d, "\sim"', "p_nG"] \\ P_nF \arrow[r, "P_n(\alpha)"'] & P_nG. \end{tikzcd}$$

G is degree n, so p_nG is an equivalence. Thus α factors through P_nF.

Another important point is that if $k \leq n$ then the map

$$P_k(p_nF) : (P_kF)(X) \longrightarrow (P_kP_nF)(X)$$

is an equivalence (for any F). To see this, it is enough to note that P_k commutes with sequential hocolim's and to verify that

$$P_k(t_nF) : (P_kF)(X) \longrightarrow (P_kT_nF)(X)$$

is an equivalence. The latter statement follows from the facts that P_k commutes with finite homotopy colimits and that the map

$$t_nP_kF : (P_k)(X) \longrightarrow (T_nP_kF)(X)$$

is an equivalence since P_k is degree k and therefore also degree n.

In other words, we can sum up the discussion as

Proposition 2. *The functor $P_nP_k(F)$ is equivalent to $P_{\min\{n,k\}}(F)$.*

2.1. The Taylor Tower

Note that since P_nF is degree n and therefore also degree $n+1$, the universal property gives a map $q_{n+1}F : P_{n+1}F \to P_nF$ such that

$$\begin{tikzcd} F \arrow[r, "p_{n+1}F"] \arrow[dr, "p_nF"'] & P_{n+1}F \arrow[d, "q_{n+1}F"] \\ & P_nF \end{tikzcd}$$

commutes. The various P_nF then assemble into a tower, which is called the **Taylor Tower** of F (based at $*$).

3. The Layers in the Taylor Tower

Given a functor F, we define a new functor D_nF to be the homotopy fiber of r_nF:

$$D_nF \to P_nF \xrightarrow{q_nF} P_{n-1}F.$$

Thus $D_1F(X)$ is the homotopy fiber of the map

$$P_1F(X) \to P_0F(X) = F(*),$$

or in other words the reduced version of the functor P_1F.

Say that a functor G is **homogeneous of degree** n if it is degree n and $P_{n-1}G \simeq *$.

Proposition 3. *The nth layer D_nF of the Taylor tower is a homogeneous degree n functor.*

The fact that D_nF is degree n follows from the next result, which is really a statement about homotopy limits commuting.

Proposition 4. *Suppose that*

$$F \longrightarrow G \longrightarrow H$$

is a fiber sequence of functors and that G and H are both degree n. Then so is F.

That $P_{n-1}D_n(F)$ is trivial follows from (1) the fact that P_{n-1} preserves fiber sequences (commutation of finite limits with sequential colimits and with limits) and (2) the fact that

$$P_{n-1}P_nF \xrightarrow{P_{n-1}q_n} P_{n-1}P_{n-1}F$$

is an equivalence.

Example 2. An example of a homogeneous degree n functor is the functor from spectra to spectra defined by $X \mapsto X^{\wedge n}$. This follows from Goodwillie's result (Calc II, Prop 3.4) that if $L : \mathscr{C}^n \to \mathscr{D}$ is degree k_i in the ith variable, then the composite functor $\mathscr{C} \xrightarrow{\Delta} \mathscr{C}^n \xrightarrow{L} \mathscr{D}$ is degree $k_1 + \cdots + k_n$.