# THE CALCULUS OF FUNCTORS: HOMOGENEOUS FUNCTORS TALK 4

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## 1. Homogeneous Functors

Recall from last time that a functor $F$ is **homogeneous of degree** $n$ if it is degree $n$ and $P_{n-1}F \simeq *$.

**Example 1.** The $n$th layer $D_nF$ of the Taylor tower is a homogeneous degree $n$ functor.

The fact that $D_nF$ is degree $n$ follows from the next result, which is really a statement about homotopy limits commuting.

**Proposition 1.** *Suppose that*

$$F \longrightarrow G \longrightarrow H$$

*is a fiber sequence of functors and that $G$ and $H$ are both degree $n$. Then so is $F$.*

That $P_{n-1}D_n(F)$ is trivial follows from (1) the fact that $P_{n-1}$ preserves fiber sequences (commutation of finite limits with sequential colimits and with limits) and (2) the fact that

$$P_{n-1}P_nF \xrightarrow{P_{n-1}q_n} P_{n-1}P_{n-1}F$$

is an equivalence.

**Example 2.** An example of a homogeneous degree $n$ functor is the functor from spectra to spectra defined by $X \mapsto X^{\wedge n}$ (or $X \mapsto \Sigma^\infty X^{\wedge n}$ from based spaces to spectra). That this functor is degree $n$ follows from Goodwillie's result (Calc II, Prop 3.4) that if $L : \mathscr{C}^n \to \mathscr{D}$ is degree $k_i$ in the $i$th variable, then the composite functor $\mathscr{C} \xrightarrow{\Delta} \mathscr{C}^n \xrightarrow{L} \mathscr{D}$ is degree $k_1 + \cdots + k_n$. That the functor is homogeneous now follows from Goodwillie's Lemma 3.1 (Calc III), which says that if $F$ is a functor of $n$ variables, reduced with respect to each variable, then $P_{n-1}(F \circ \Delta) \simeq *$. More generally, given any spectrum $C$, the functor

$$X \mapsto C \wedge X^{\wedge n}$$

is homogeneous of degree $n$.

**Example 3.** Suppose that $F : \mathscr{C} \longrightarrow \mathscr{D}$ is homogeneous of degree $k$, $\mathscr{D}$ is stable, and $F$ factors through the category $\Sigma_n - \mathscr{D}$ of objects with a $\Sigma_n$-action. Again, $X \mapsto X^{\wedge n}$ would be an example. Then the composite functor $F(X)_{h\Sigma_n}$ is again homogenous. The point is that (1) since $\mathscr{D}$ is stable, degree $k$ functors are those taking strongly cocartesian $k+1$-cubes in $\mathscr{C}$ to cocartesian $k+1$-cubes in $\mathscr{D}$ and (2) homotopy orbits with respect to $\Sigma_n$ is a homotopy colimit and will thus preserve cocartesian cubes. More generally, given any spectrum $C$ with a $\Sigma_n$-action, the functor

$$X \mapsto (C \wedge X^{\wedge n})_{h\Sigma_n}$$

is homogeneous of degree $n$.

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The description of homotopy orbits as a homotopy colimit is as follows. Consider the category $\mathscr{S}_n$ having a single object $\star$ and $\mathrm{End}(\star) = \Sigma_n$. Then the category of $\Sigma_n$-objects in $\mathscr{D}$ is equivalent to the category of functors $\mathscr{S}_n \longrightarrow \mathscr{D}$. Thus a $\Sigma_n$-object $X$ in $\mathscr{D}$ corresponds to a functor $\mathbb{X} : \mathscr{S}_n \longrightarrow \mathscr{D}$, and there is an identification $X_{h\Sigma_n} \simeq \mathrm{hocolim}_{\mathscr{S}_n} \mathbb{X}$.

## 2. The formula for $D_n(F)$

Since $D_nF(X)$ captures precisely the degree $n$ part of the Taylor tower and not the lower degree information, we think of it as an analogue of $\frac{f^{(n)}(0)}{n!}x^n$. It turns out that there is in fact a formula for $D_nF(X)$ that resembles the formula from the calculus of functions.

### 2.1. $n = 1$

Let's begin with $D_1(F)(X)$ since we already know $D_0(F)(X) = P_0(F)(X)$ is a constant. Let us assume for simplicity that $F$ is a functor from based spaces to spectra. Define a spectrum $\partial^{(1)}(F)(*)$, called the **first derivative** of $F$ at $*$, to be $D_1(F)(S^0)$. This new spectrum plays the role of $f'(0)$. Then, since $D_1(F)$ takes homotopy pushouts of spaces to homotopy pushouts of spectra (homotopy pushout squares of spectra are the same are homotopy pullback squares), we conclude that for any *finite* complex $K$ we have an equivalence

$$\partial^{(1)}(F)(*) \wedge K \simeq D_1(F)(K).$$

If we want such an equivalence for all spaces $K$, then we need a further assumption that $D_1(F)$ preserves filtered homotopy colimits (such functors are called **finitary**). It turns out that assuming that $F$ satisfies this property implies the same for $D_1(F)$.

**Remark 1.** An example of a degree one functor that is *not* finitary can be given as follows. Choose an infinite complex $W$ and define a functor $F_W$ by

$$F_W(X) = \mathbb{M}\mathrm{ap}(\Sigma^\infty W_+, \Sigma^\infty X),$$

where $\mathbb{M}\mathrm{ap}$ denotes the mapping spectrum. There is a canonical map

$$D(\Sigma^\infty W_+) \wedge X \longrightarrow F_W(X),$$

where $D$ denotes the Spanier-Whitehead dual, but this map is not an equivalence if neither $W$ nor $X$ is finite. Thus $F_W$ is not finitary.

**Example 4.** Recall that we previously considered the four functors

$$\mathrm{Id}_{\mathbf{Top}_*} : \mathbf{Top}_* \longrightarrow \mathbf{Top}_*,$$

$$\Sigma^\infty : \mathbf{Top}_* \longrightarrow \mathbf{Sp},$$

$$\Omega^\infty : \mathbf{Sp} \longrightarrow \mathbf{Top}_*,$$

and

$$\mathrm{Id}_{\mathbf{Sp}} : \mathbf{Sp} \longrightarrow \mathbf{Sp}.$$

All except the first are homogeneous degree 1 functors (and all are finitary). Although we have really only discussed the first derivatives of functors landing in spectra, one can also do this for functors with general codomain, and it turns out that the first derivative of all four of these functors is the sphere spectrum $\mathbb{S}^0$.

### 2.2. $n > 1$

To obtain the formula for $D_n(F)$ for $n > 1$, we will need Goodwillie's result (Calc III, Theorem 3.5) that there is a bijection

$$\left\{ \begin{array}{l} \text{symmetric multilinear} \\ \text{functors } L : \mathscr{C}^n \longrightarrow \mathscr{D} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{homogeneous degree } n \\ \text{functors } F : \mathscr{C} \longrightarrow \mathscr{D} \end{array} \right\}.$$

To a symmetric, multilinear $L$ is associated the homogenous functor $(L \circ \Delta)_{h\Sigma_n}$. To get a symmetric, multilinear functor from a homogeneous one, we need the notion of the "cross effect of a functor".

The $n$th cross effect of a functor $F$ is a functor of $n$ variables that measures the failure of $F$ to be a degree $n-1$ functor. For instance, the first cross effect $\mathrm{cr}_1(F)$ is the fiber

$$\mathrm{cr}_1(F)(X) \longrightarrow F(X) \longrightarrow F(*),$$

which is trivial if $F$ is degree 0 (constant). The second cross effect is defined using the cocartesian square

$$\begin{tikzcd} X \vee Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & *. \end{tikzcd}$$

Let us write $\beta_0(X, Y)$ for the diagram obtained by removing the vertex $X \vee Y$. The second cross effect is then defined to be the fiber

$$\mathrm{cr}_2(F)(X, Y) \longrightarrow F(X \vee Y) \longrightarrow \operatorname{holim} F(\beta_0(X, Y)).$$

More generally, one defines $\mathrm{cr}_n(F)$ by a fiber sequence

$$\mathrm{cr}_n(F)(X_1, \dots, X_n) \longrightarrow F(X_1 \vee \cdots \vee X_n) \longrightarrow \operatorname{holim} F(\beta_0(X_1, \dots, X_n)).$$

The cross effect is symmetric since it is defined as the fiber of a map of symmetric functors, and the $n$th cross effect of a degree $n$ functor is multilinear by Calc III, Prop. 3.3.

The functor $\mathrm{cr}_n D_n(F)$ is then a symmetric multilinear functor (for any $F$). It is denoted $D^{(n)}(F)$ and called the $n$-fold differential of $F$. We now define the $n$th derivative of $F$ at the basepoint $*$ to be

$$\partial^{(n)} F(*) := D^{(n)} F(S^0, \dots, S^0).$$

Multilinearity now gives that

$$D^{(n)} F(X_1, \dots, X_n) \simeq \partial^{(n)} F(*) \wedge X_1 \wedge \ldots \wedge X_n$$

if the $X_i$'s are finite or more generally if $F$ is finitary.

According to the correspondence between homogeneous functors and symmetric multilinear ones, we have $D_n F \simeq (D^{(n)}(F) \circ \Delta)_{h\Sigma_n}$, so that we conclude

$$D_n F(X) \simeq (\partial^{(n)} F(*) \wedge X^{\wedge n})_{h\Sigma_n}$$

for finite $X$ or for all $X$ in the finitary case. This is the desired formula for the $n$th layer in the Taylor tower.

On the other hand, in order to calculate $D_n(F)(X)$ using this formula, one must first find $\partial^{(n)}(F)(*)$, which is defined using the cross effects of $D_n(F)$. It would be great to be able to understand $\partial^{(n)}(F)(*)$ without already knowing the functor $D_n(F)$. The following result of Goodwillie allows us to do this.

**Theorem 1** (Calc III, Theorem 6.1)**.** *The $n$th differential $D^{(n)}(F)$ is equivalent to the multilinearization of the $n$th cross effect of $F$.*

The multilinearization of a functor $G$ of $n$ variables is the effect of linearizing $G$ with respect to each variable. For instance, for a functor $G$ of two variables with is reduced with respect to both variables, the multilinearizaiton would be the functor $\operatorname{hocolim}_{n,k} \Omega^n \Omega^k G(\Sigma^n X, \Sigma^k Y)$.