Computation of $H^*(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$

As the title suggests – our goal is to compute mod 2 cohomology of $K(\mathbb{Z}_2, n)$. The computation of the cohomology of the Eilenberg-MacLane spaces will be performed via induction on the connectivity of the space. The inductive step will be performed using Serre spectral sequence of the fiber sequence $K(\mathbb{Z}_2, n) \longrightarrow * \longrightarrow K(\mathbb{Z}_2, n+1)$. However, instead of tackling the problem directly, we will discuss some general properties of Serre spectral sequence and then apply those to our situation.

Suppose we are given a fibration $F \longrightarrow X \longrightarrow B$, and that it satisfies the usual conditions for existence of properly convergent Serre spectral sequence, i.e. the trivial action of $\pi_1(B, *)$ on $H^*(F)$. Set the coefficient group to be G. Let us look at the E_n-page of the spectral sequence for cohomology, where $n \geq 2$; there we have a differential $d^n : E_n^{0,n-1} \longrightarrow E_n^{n,0}$. This differential is often referred to as *transgression*. There is one transgression per dimension; therefore, we will write it as τ, since most of the time the dimension is understood. Note that $E_n^{0,n-1}$ is a subgroup of $E_2^{0,n-1} = H^0(B; H^{n-1}(F; G)) = H^{n-1}(F; G)$, since no non-trivial differentials enter $E_*^{0,n-1}$ spot of the spectral sequence. Similarly, $E_n^{n,0}$ is a quotient of $H^n(B; G)$. We can try to "manufacture" a map relating $H^{n-1}(F; G)$ to $H^n(B; G)$, using ordinary long exact sequence coboundary map. Here is what we will eventually arrive at:

$$
\begin{tikzcd}
H^{n-1}(F;G) \arrow[r, "\delta"] & H^n(X,F;G) & H^n(B,*;G) \arrow[l, "p^*"'] \arrow[r, "j^*", "\cong"'] & H^n(B;G) \arrow[d] \\
\delta^{-1}(\operatorname{im} p^*) \arrow[u, hook] \arrow[rrr, "\mu"] & & & H^n(B;G)/j^*(\ker p^*)
\end{tikzcd}
$$

We get a map from $\delta^{-1}(\operatorname{im} p^*)$ to $H^n(B; G)/j^*(\ker p^*)$. Note also that j^* is an isomorphism if $n > 0$. It is not unnatural to expect $\delta^{-1}(\operatorname{im} p^*) = E_n^{0,n-1} \subset E_2^{0,n-1} = H^{n-1}(F; G)$ and $H^n(B; G)/j^*(\ker p^*) \cong E_n^{n,0}$ and the map μ be the same as τ. In fact, the following statement is true.

Proposition. *There exists a commutative diagram,*

$$
\begin{tikzcd}
\delta^{-1}(\operatorname{im} p^*) \arrow[r, "\mu"] \arrow[d, "i"', "\cong"] & H^n(B;G)/j^*(\ker p^*) \arrow[d, "\cong"', "\nu"] \\
E_n^{0,n-1} \arrow[r, "\tau"] & E_n^{n,0}
\end{tikzcd}
$$

Proof. See [Hatcher, SSAT, Ch1, pp.21-22]. ■

We set our coefficient group to be $\mathbb{Z}_2$ from now on. The action of Steenrod squares on $H^*(F; \mathbb{Z}_2)$ and $H^*(B; \mathbb{Z}_2)$ descend to an action on $E_n^{0,n-1}$ and $E_n^{n,0}$. The elements of $E_n^{0,n-1}$ in $H^*(F; \mathbb{Z}_2)$ are called *transgressive*. If α is transgressive then $\delta\alpha = p^*\beta$ for some $\beta \in H^n(B, *; \mathbb{Z}_2)$. Let Sq be a Steenrod square. Then $\delta \mathrm{Sq}\,\alpha = \mathrm{Sq}\,\delta\alpha = \mathrm{Sq}\,p^*\beta = p^*\mathrm{Sq}\,\beta$, implying that $\mathrm{Sq}\,\alpha$ is also transgressive. Let us emphasize that the coboundary operations and squares commute due to stability of squares. Similarly, an element of $j^*(\ker p^*)$ is form, $j^*(\gamma)$, where $p^*\gamma = 0$. Then $\mathrm{Sq}\,j^*(\gamma) = j^*(\mathrm{Sq}\,\gamma)$ and $p^*\mathrm{Sq}\,\gamma = \mathrm{Sq}\,p^*\gamma = 0$. Thus, we have a well-defined action of Steenrod squares on $E_n^{n,0}$. Furthermore, we claim that squares commute with transgressions. Indeed, if $\delta\alpha = p^*\beta$, then $\overline{j^*\beta} = \tau\alpha$. Then $\delta\mathrm{Sq}\,\alpha = p^*\mathrm{Sq}\,\beta$, i.e. $\tau\mathrm{Sq}\,\alpha = \overline{j^*\mathrm{Sq}\,\beta} = \overline{\mathrm{Sq}\,j^*\beta} = \mathrm{Sq}\,\overline{j^*\beta} = \mathrm{Sq}\,\tau\alpha$. Thus, we can see that transgression work well with Steenrod squares.

In certain special situations transgressions can give enough information to compute $H^*(B; \mathbb{Z}_2)$ in terms of $H^*(\Omega B; \mathbb{Z}_2)$. The following theorem is due to Borel.

Theorem. *Let B be a simply-connected space and $H^*(\Omega B; \mathbb{Z}_2)$ have a simple system of transgressive generators, with finitely many of those generators in each dimension. Then $H^*(B; \mathbb{Z}_2)$ is a polynomial ring generated by (any) representatives of transgressions of Serre spectral sequence for the fiber sequence $\Omega B \longrightarrow * \longrightarrow B$, of this simple generators.*

If k is a field, a simple system of generators for a k-algebra A, is a subset S of A, such that the elements of form $x_1 x_2 \ldots x_n$, where x_i's are distinct elements of S, form a basis for A as a k-module.

Proof. See [Mosher/Tangora, Ch9, pp.91-92] & [Hatcher, SSAT, Ch1, pp.54-58]. ■

Note that a graded polynomial algebra $\mathbb{Z}_2[x_1, x_2, \dots]$ has a simple system of generators, namely $\{x_i^{2^j}\}$. Thus, if we have that $H^*(\Omega B; \mathbb{Z}_2)$ is a polynomial algebra over a set of trangressive elements, then so is $H^*(B; \mathbb{Z}_2)$. This is a consequence of the fact that $x^{2^j} = \mathrm{Sq}^{2^{j-1}n} \dots \mathrm{Sq}^{2n} \mathrm{Sq}^{n} x$ (n is the degree of x) and that the squares preserve transgressiveness. This is a good place to transition to a more concrete situation.

We begin with $K(\mathbb{Z}_2, 1)$. We know that $K(\mathbb{Z}_2, 1) = \mathbb{R}P^\infty$. Using the standard cell structure of $\mathbb{R}P^\infty$ or the Gysin sequence of the universal line bundle $L \longrightarrow \mathbb{R}P^\infty$, we can determine that $H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) = \mathbb{Z}_2[\iota_1]$, where ι_1 is the non-zero element of dimension 1 (and hence is the fundamental class). Furthermore, ι_1 is transgressive, since the differentials before τ all map into a zero group. Thus, using the previous observation $H^*(K(\mathbb{Z}_2, 2); \mathbb{Z}_2)$ is polynomial algebra over $\mathbb{Z}_2$. In fact, we can continue the argument, and understand what the generators of this cohomology ring are using Borel's theorem. The following theorem is due to Serre.

Theorem. *$H^*(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$ is the polynomial ring $\mathbb{Z}_2[Sq^I(\iota_n)]$, where ι_n is the generator of $H^n(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$ and I ranges over all admissible sequences of excess $e(I) < n$.*

As a reminder a finite sequence $I = (i_1, i_2, ...)$ is admissible if $i_j \geq 2 i_{j+1}$, $\mathrm{Sq}^I x = \mathrm{Sq}^{i_1} \mathrm{Sq}^{i_2} ... x$, and $e(I) = i_1 - \sum_{j \geq 2} i_j$. The theorem will follow directly from the following easy observations.

Lemma. *(a) $Sq^I(\iota_n) = 0$ if I is admissible and $e(I) > n$.*

(b) The elements $Sq^I(\iota_n)$ with I admissible and $e(I) = n$ are exactly the powers $(Sq^J(\iota_n))^{2^j}$ with J admissible, $e(J) < n$, and $j > 0$.

Proof. (a) Note that $i_1 = e(I) + \sum_{j \geq 2} i_j > n + \sum_{j \geq 2} i_j = |\mathrm{Sq}^{i_2} \mathrm{Sq}^{i_3} \dots (\iota_n)|$, which implies that $\mathrm{Sq}^I(\iota_n) = \mathrm{Sq}^{i_1} \mathrm{Sq}^{i_2} \dots (\iota_n) = 0$.

(b) Let us pick $\mathrm{Sq}^I(\iota_n)$, such that $e(I) = n$. Then $i_1 = n + \sum_{j \geq 2} i_j = |\mathrm{Sq}^{\widetilde{I}}(\iota_n)|$, where $\widetilde{I} = (i_2, i_3, \dots)$. Using the squaring property we get $\mathrm{Sq}^I(\iota_n) = \mathrm{Sq}^{i_1} \mathrm{Sq}^{\widetilde{I}}(\iota_n) = (\mathrm{Sq}^{\widetilde{I}}(\iota_n))^2$. The sequence $\widetilde{I}$ is clearly admissible, and $e(\widetilde{I}) = i_2 - \sum_{j \geq 3} i_j = e(I) + (2i_2 - i_1) \leq e(I) = n$. If $e(\widetilde{I}) = n$, we repeat the process, otherwise – we stop. The process will eventually stop, yielding an equality of form $\mathrm{Sq}^I(\iota_n) = (\mathrm{Sq}^J(\iota_n))^{2^j}$, where $e(J) < n$ and $j > 0$, which is what we want.

To prove the converse inclusion, we essentially reverse the argument. Suppose we are given an element $(\mathrm{Sq}^J(\iota_n))^{2^j}$, where J is admissible, $e(J) \leq n$ and $j > 0$. Let $\widetilde{J}$ be the same as J except with an extra $n + |J|$ on its left, where $|J|$ is the degree of J. Clearly, $e(\widetilde{J}) = n$. Note that $n + |J| - 2j_1 = n - j_1 + \sum_{i \geq 2} j_i = n - e(J) \geq 0$, which implies that $\widetilde{J}$ is admissible. We continue the process until we reach the equality $(\mathrm{Sq}^J(\iota_n))^{2^j} = \mathrm{Sq}^I(\iota_n)$, where I is admissible and $e(I) = n$. ■

Proof of Serre's theorem. We addressed the $n = 1$ case earlier. So suppose the statement of the theorem is true for n. Then $H^*(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$ has a simple system of generators $\{(\mathrm{Sq}^J(\iota_n))^{2^j}\}$, such that $e(J) < n$. ι_n is trangressive for the same reason that ι_1 was. Therefore, the simple system of generators we have consists of transgressive elements. This set is the same as $\mathrm{Sq}^I(\iota_n)$, where $e(I) \leq n$. Thus, $H^*(K(\mathbb{Z}_2, n + 1); \mathbb{Z}_2)$ is generated by representatives of elements $\tau(\mathrm{Sq}^I(\iota_n)) = \mathrm{Sq}^I(\tau(\iota_n)) = \mathrm{Sq}^I(\overline{\iota_{n+1}}) = \overline{\mathrm{Sq}^I(\iota_{n+1})}$, specifically, $\mathrm{Sq}^I(\iota_{n+1})$, where I is admissible and $e(I) < n + 1$. This completes the inductive step and the proof of the theorem. ■

Finally, let us mention what implications this theorem has on the structure of Steenrod algebra. We remind the reader that the Steenrod algebra, written as $\mathcal{A}_2$, is the $\mathbb{Z}_2$-algebra of stable reduced cohomology operations. Here is a precise definition. First we consider general operations, which we define as natural transformations $\widetilde{H}^k(-; \mathbb{Z}_2) \longrightarrow \widetilde{H}^{n+k}(-; \mathbb{Z}_2)$ for various $k, n \in \mathbb{N}$. Let this set be denoted by $\mathcal{O}_{n,k}$. Clearly, $\mathcal{O}_{n,k}$ is a $\mathbb{Z}_2$-module. Assemble these modules into a bigger one $\mathcal{O}_n = \bigoplus_{k=1}^\infty \mathcal{O}_{n,k}$. Consider all the elements of form $\eta - \zeta \in \mathcal{O}_n$, where $\eta \in \mathcal{O}_{n,k}$ and $\zeta \in \mathcal{O}_{n,k+r}$, such that for any cohomology class α of degree k, $\Sigma^r \eta(\alpha) = \zeta(\Sigma^r \alpha)$. These elements form a $\mathbb{Z}_2$-submodule. Let the quotient module of $\mathcal{O}_n$ by this submodule be denoted by $\mathcal{S}_n$. Define $\mathcal{A}_2$ to be $\bigoplus_{n=0}^\infty \mathcal{S}_n$. The algebra structure is given by composition.

Let η be an operation in $\mathcal{O}_{n,k}$. Using Yoneda type argument one can show that η is completely determined by its value at the fundamental class $\eta(\iota_k) \in \widetilde{H}^{n+k}(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$. By Serre's theorem, $\eta(\iota_k)$ is a polynomial $p\left(Sq^I(\iota_k)\right)$, where I ranges over admissible sequences with excesses less than n, and p does not contain any degree 0 terms. By naturality of Steenrod squares and cup product, we can conclude that $\eta(\alpha) = p(\mathrm{Sq}^I(\alpha))$. Define another operation $\widehat{\eta} \in \mathcal{O}_{n,k+1}$ via the equation $\widehat{\eta}(\alpha) = \widehat{p}\left(Sq^I(\alpha)\right)$, where $\widehat{p}$ is the linear part of p. The elements η and $\widehat{\eta}$ represent the same class in $\mathcal{A}_2$: $\Sigma \eta(\alpha) = \Sigma p(\mathrm{Sq}^I(\alpha)) = \Sigma \widehat{p}(\mathrm{Sq}^I(\alpha)) = \widehat{p}(\mathrm{Sq}^I(\Sigma \alpha)) = \widehat{\eta}(\Sigma \alpha)$. The fact that only the linear part of the polynomial remains is a consequence of the fact that the cup product on suspended spaces is 0. Thus, any element of $\mathcal{A}_2$ is represented by a linear polynomial on admissible sequences of Steenrod operations. Now suppose $\eta \in \mathcal{O}_{n,k}$ and $\zeta \in \mathcal{O}_{n,k+r}$ represent the same element in $\mathcal{A}_2$, and $\eta(\alpha) = p(\mathrm{Sq}^I(\alpha))$ and $\zeta(\beta) = q(\mathrm{Sq}^I(\beta))$, where p and q are linear. Then $\Sigma^r p(\mathrm{Sq}^I(\iota_k)) = \Sigma^r \eta(\iota_k) = \zeta(\Sigma^r \iota_k) = q(\mathrm{Sq}^I(\Sigma^r \iota_k)) = \Sigma^r q(\mathrm{Sq}^I(\iota_k))$, which implies $p = q$, since both of the polynomials are linear. Thus, any element of Steenrod algebra can be *uniquely* expressed as a $\mathbb{Z}_2$-linear combination of admissible Steenrod squares. However, we know that using Adem relations one can convert any sequence of squares into an a sum of admissibles. Therefore, the following theorem holds.

THEOREM. *$\mathcal{A}_2$ is the free associative algebra on symbols $\{Sq^n\}_{n=0}^{\infty}$ modulo the Adem relations.* ■