Summer Seminar Notes:
Taylor Tower of the Forgetful Functor

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Introduction

Let k be a commutative (probably simplicial) ring (but not yet ring spectrum). We'll be working with functors between the two categories:

$$\mathscr{C} := k\backslash\mathrm{CommRings}/k \longrightarrow k\text{-Modules} =: \mathscr{D}$$

Remark on stability: I need to pull the stability card in one place, I'll mention it explicitly, and what this means is that we're working with chain complexes of k-Modules. Honestly, that's where we should be working if I'm going to be faithful to [2], but I'm trying to prep you to think about [5] and [4].

Let U denote the forgetful functor from $\mathscr{C}$ to $\mathscr{D}$ and $\widetilde{X}$:=the cofibrant-fibrant replacement of X. We will prove the following:

Theorem 0.1.

$$P_1U(X) \simeq I/I^2(\widetilde{X})$$

i.e. P_1U *is the derived functor of* I/I^2.

Theorem 0.2.

$$P_nU(X) \simeq I/I^{n+1}(\widetilde{X})$$

i.e. P_nU *is the derived functor of* I/I^{n+1}.

Corollary 0.3.

$$D_nU(X) \simeq I^n/I^{n+1}(\widetilde{X})$$

CAVEAT: I have not been faithful to the forms of the proofs in [2]. The form included here of the proof for Theorem 0.1 follows from talks I had with Kristine Bauer this summer. The lemma used to prove 0.2 is as in [2], but the proof is a bit different otherwise. They constructed the D_n's first and used a 5-/snake-lemma to conclude with induction what they wanted.

Theorem 0.1

We're first going to reduce our calculation to that of the augmentation ideal I.

Given $X \in \mathscr{C}$, X is of the form $k \oplus I(X)$ where $I(X) = ker(X \to k)$.

Remark 0.4. *U is not a reduced functor since $U(k) = k \neq 0$.*

$$\widetilde{U}(X) := \frac{U(X)}{U(k)} = \frac{I(X) \oplus k}{k} \simeq I(X)$$

Recall that to construct P_1 (via the cotriple model), we need to calculate cross-effects, and $cr_2 F \simeq cr_2 \widetilde{F}$ (reduced). So we might as well use $\widetilde{U} = I$.

Therefore, by Remark 0.4, constructing the Taylor Tower of U boils down to constructing that of I.

Proof of Theorem 0.1.

Remark 0.5. *For $X \in \mathscr{C}$,*

$$X \otimes_k X \simeq k \oplus I \oplus I \oplus (I \otimes I)$$

Let's calculate $cr_2\widetilde{U}(X)$. This is given as

$$\begin{aligned} \text{total hofib} \begin{pmatrix} \widetilde{U}(X \otimes X) & \longrightarrow & \widetilde{U}(X) \\ \downarrow & & \downarrow \\ \widetilde{U}(X) & \longrightarrow & 0 \end{pmatrix} &\simeq \text{total hofib} \begin{pmatrix} I \oplus I \oplus (I \otimes I) & \longrightarrow & I \\ \downarrow & & \downarrow \\ I & \longrightarrow & 0 \end{pmatrix} \\ &\simeq \text{hofib}(I \oplus I \oplus (I \otimes I) \to I \oplus I) \\ &\ \text{(Since the hoPB is } I \oplus I\text{, as modules)} \\ &\simeq I \otimes_k I \end{aligned}$$

Then

$$\begin{aligned} P_1\widetilde{U}(X) &\simeq \text{hocof}(cr_2\widetilde{U}(X) \xrightarrow{+} \widetilde{U}(X)) \\ &\simeq \text{hocof}(I(X)\otimes I(X) \xrightarrow{m} I(X)) \\ &\simeq I/I^2(\widetilde{X}) \end{aligned}$$

$\square$

Theorem 0.2 and Corollary 0.3

First we need the following Lemma:

Lemma 0.6.

$$cr_n I \simeq I^{\otimes_k n}$$

Recall the inductive definition of cross-effect:

Definition 0.7 (due to [1])**.** *In our setting, the nth cross-effect of F is the functor defined inductively by*

$$cr_nF(M_1,\dots,M_n) \oplus cr_{n-1}F(M_1,M_3,\dots,M_n) \oplus cr_{n-1}F(M_2,M_3,\dots,M_n) \cong cr_{n-1}F(M_1\oplus M_2,M_3,\dots,M_n)$$

that is,

$$cr_nF(M_1,\dots,M_n) \cong \frac{cr_{n-1}F(M_1\otimes M_2,M_3,\dots,M_n)}{cr_{n-1}F(M_1,M_3,\dots,M_n)\oplus cr_{n-1}F(M_2,M_3,\dots,M_n)}$$

Proof of Lemma 0.6. By induction. Previous work in the proof of Theorem 0.1 establishes our (first interesting) base case, that $cr_2 I \simeq I\otimes I$. Now assume $cr_{n-1}I \simeq I^{\otimes_k (n-1)}$.

Then,

$$\begin{aligned} cr_nI(X_1,\dots,X_n) &\cong \frac{cr_{n-1}I(X_1\otimes_k X_2,X_3,\dots,X_n)}{cr_{n-1}I(X_1,X_3,\dots,X_n)\oplus cr_{n-1}I(X_2,X_3,\dots,X_n)} && \text{by Definition 0.7} \\ &\cong \frac{I(X_1\otimes_k X_2)\otimes_k I(X_3)\otimes_k\cdots\otimes_k I(X_n)}{(I(X_1)\otimes_k I(X_3)\otimes_k\cdots\otimes_k I(X_n))\oplus(I(X_2)\otimes_k\cdots\otimes_k I(X_n))} && \text{by inductive hyp} \\ &\cong \frac{[I(X_1)\oplus I(X_2)\oplus(I(X_1)\otimes_k I(X_2))]\otimes_k I(X_3)\otimes_k\cdots\otimes_k I(X_n)}{(I(X_1)\otimes_k I(X_3)\otimes_k\cdots\otimes_k I(X_n))\oplus(I(X_2)\otimes_k\cdots\otimes_k I(X_n))} && \text{breaking up } I(X_1\otimes_k X_2) \\ &\cong I(X_1)\otimes_k\cdots\otimes_k I(X_n) \end{aligned}$$

Therefore, $cr_n\circ\text{diag}(X)\simeq I^{\otimes_k n}(X)$.

$\square$

Now we can prove Theorem 0.2 quite simply.

Proof of Theorem 0.2.

$$\begin{aligned} P_n\widetilde{U}(X) &\simeq \text{hocof}(cr_n I(X) \to I(X)) \\ &\simeq \text{hocof}(I^{\otimes_k (n+1)} \to I(X)) \quad \text{by Lemma 0.6} \\ &\simeq I/I^{n+1}(\widetilde{X}) \end{aligned}$$

□

The following holds for any pointed model category, not necessarily one that is stable. We're working in an augmented setting, which counts as 'pointed'. This is the version I learned from Tom Goodwillie, but I think it is common, as in [3].

Theorem 0.8 (Octahedral Axiom). *Given a sequence of maps $A \to B \to C$, the following is a cofiber sequence*

$$cof(A \to B) \to cof(A \to C) \to cof(B \to C)$$

where we usually mean 'homotopy cofiber' when we say 'cofiber'. By cofiber sequence, I mean that $cof(B \to C) = cof[cof(A \to B) \to cof(A \to C)]$.

Proof of Corollary 0.3. Let's apply the octahedral axiom to the following sequence of k-modules

$$I^{n+1}(X) \to I^n(X) \to I(X)$$

This yields the following cofiber sequence

$$\begin{array}{ccccc} \text{cof}(I^{n+1}(X) \to I^n(X)) & \longrightarrow & \text{cof}(I^{n+1}(X) \to I(X)) & \longrightarrow & \text{cof}(I^n(X) \to I(X)) \\ \cong & & \cong & & \cong \\ I^{n+1}/I^n(X) & \longrightarrow & I/I^{n+1}(X) & \longrightarrow & I/I^n(X) \end{array}$$

Theorem 0.2 let's us rewrite this as

$$I^{n+1}/I^n(X) \longrightarrow P_n I(X) \longrightarrow P_{n-1} I(X) \qquad (*)$$

and here's where we need stability.

A note on stability: in the (anticipated) 'Brave New'/E_∞ setting, $k-$Modules will be stable. Here, to make everything play nicely, since I'm acting 'classically', I need to mention that we're landing in *chain complexes* of k-Modules, so that I can then say that $(*)$ is not only a cofiber sequence but a *fiber* sequence, i.e. $I^{n+1}/I^n(X) = \text{hofib}(P_n I(X) \to P_{n-1} I(X))$, where we know that by definition, $D_n I(X) = \text{hofib}(P_n I(X) \to P_{n-1} I(X))$, so we have our desired result. □

References

[1] Samuel Eilenberg and Saunders MacLane, On the groups $h(\pi, n)$, ii: Methods of computation, The Annals of Mathematics, Second Series **60** (1954), no. 1, 49–139.

[2] M Kantorovitz and R McCarthy, The taylor towers for rational algebraic K-theory and hochschild homology, Homology, Homotopy and Applications **4** (2002), 191–212.

[3] Jason Lo, Octahedral axiom in triangulated categories, Notes from talk in Student Algebraic Geometry Seminar at Stanford, `http://math.stanford.edu/~jasonlo/writings/octahedral_talk.pdf`, Nov 2008.

[4] R McCarthy and V Minasian, On triples, operads and generalized homogeneous functors, Preprint (2004).

[5] Randy McCarthy and Vahagn Minasian, Hkr theorem for smooth s-algebras, Journal of Pure and Applied Algebra **185** (2003), no. 1-3, 239 – 258.