A SHORT NOTE ON MODELS FOR EQUIVARIANT HOMOTOPY THEORY

B. GUILLOU

1. Introduction

These notes explore equivariant homotopy theory from the perspective of model categories in the case of a discrete group G. Section 2 reviews the situation for topological spaces, largely following [May]. In section 3, we discuss two approaches to equivariant homotopy theory in more general model categories. Section 4 discusses some examples to which the material from Section 3 applies. In particular, the final example discusses equivariant homotopy theory in Morel and Voevodsky's $\mathbb{A}^1$-homotopy category (still for a discrete group G). Finally, section 5 discusses briefly the coarse, or naive, model structure.

2. Topological spaces

Fix a (discrete) group G. The category of spaces will be denoted by $\mathscr{T}$.

There are (at least) two approaches to equivariant homotopy theory. The first starts with the category of G-spaces $G\mathscr{T}$; that is, spaces X equipped with an action $G \times X \to X$. A G-homotopy between G-maps $X \rightrightarrows Y$ is a G-map $X \times I \to Y$, where G acts on $X \times I$ by $g(x,t) = (gx,t)$. Passage to homotopy classes yields a homotopy category $hG\mathscr{T}$ (following the notation of [May]).

As in the nonequivariant case, the above category is not the correct homotopy category; one gets the appropriate category by restricting to the subcategory of $G-CW$ complexes. Alternatively, one defines a weak equivalence of G-spaces to be a G-map $f : X \to Y$ which induces a weak equivalence on fixed point spaces $f^H : X^H \xrightarrow{\sim} Y^H$ for *every* subgroup $H \leq G$. The category obtained from $hG\mathscr{T}$ by formally inverting the weak equivalences is denoted $\bar{h}G\mathscr{T}$; this is the desired homotopy category.

On the other hand, one can start instead with the orbit category $\mathscr{O}_G$. It has objects G/H indexed by the subgroups $H \leq G$, and morphisms $G/H \to G/K$ are simply maps of G-sets (and these correspond to $\gamma \in G$ such that $\gamma H \gamma^{-1} \leq K$). Note that a G-space X determines a *contravariant* functor $X^{(-)} : \mathscr{O}_G \to \mathscr{T}$. It is defined on objects by $X^{G/H} := X^H$; given a morphism $G/H \to G/K$ induced by $\gamma \in G$, the map $X^K \to X^H$ is given by $x \mapsto \gamma \cdot x$. Note that a G-map $X \to Y$ is a weak equivalence if and only if the corresponding natural transformation $X^{(-)} \to Y^{(-)}$ is an objectwise weak equivalence. This motivates the following definition:

Definition 2.1. Given two functors $X, Y : \mathscr{O}_G^{op} \to \mathscr{T}$, a natural transformation $f : X \to Y$ is called a **weak equivalence** if it is an objectwise equivalence.

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One may then form the homotopy category $h\mathscr{T}^{\mathscr{O}_G^{op}}$ of $\mathscr{O}_G$-shaped diagrams in $\mathscr{T}$ by formally inverting the weak equivalences. As we have seen, we have a functor $\Phi : X \mapsto X^{(-)}$ from G-spaces to $\mathscr{O}_G$-spaces, and this preserves weak equivalences. Thus we get an induced functor

$$\Phi : \overline{h}G\mathscr{T} \to h\mathscr{T}^{\mathscr{O}_G^{op}}.$$

We introduced this section by saying that we would be discussing two approaches to equivariant homotopy theory, so these two homotopy categories had better be equivalent.

Theorem 2.2 (Elmendorf's Theorem; [May], Theorem VI.6.3)**.** *The functor*

$$\Phi : \overline{h}G\mathscr{T} \to h\mathscr{T}^{\mathscr{O}_G^{op}}$$

is an equivalence of categories.

In the next section, we will duplicate the previous discussion, replacing $\mathscr{T}$ by an arbitrary *cofibrantly generated model category.*

3. Model structures

In this section, $\mathscr{C}$ will be a cofibrantly generated model category with generating cofibrations I and generating acyclic cofibrations J. We will recall the definition below, but first we need to recall the notion of a small object.

Definition 3.1. An object $X \in \mathscr{C}$ is said to be **small** with respect to a subcategory $\mathscr{D}$ of $\mathscr{C}$ if there exists some cardinal κ such that for every regular cardinal[1] $\lambda \geq \kappa$ and every λ-sequence Z_* in $\mathscr{D}$ the natural map

$$\operatorname*{colim}_{\beta<\lambda} \mathrm{Hom}(X, Z_\beta) \to \mathrm{Hom}(X, \operatorname*{colim}_{\beta<\lambda} Z_\beta)$$

is an isomorphism.

Definition 3.2. A **cofibrantly generated** model category $\mathscr{C}$ is a model category such that there exists

(1) a set I of morphisms in $\mathscr{C}$ such that the domains of elements of I are small with respect to the relative I-cell complexes and such that a map is an acyclic fibration if and only if it satisfies the right lifting property (RLP) with respect to I.
(2) a set J of morphisms in $\mathscr{C}$ such that the domains of elements of J are small with respect to the relative J-cell complexes and such that a map is a fibration if and only if it satisfies the RLP with respect to I.

3.1. $\mathscr{O}_G$-shaped Diagrams in $\mathscr{C}$

We may regard $\mathscr{C}$ as being tensored over $\mathscr{O}_G$ in the following sense: for any $X \in \mathscr{C}$ and any orbit G/H, we define

$$G/H \times X := \coprod_{G/H} X.$$

[1]Assuming the Axiom of Choice, a cardinal κ is said to be **regular** if given any set X of cardinality κ and a decomposition $X \cong \amalg_{\alpha\in A} X_\alpha$ with each $|X_\alpha| < \kappa$, then $|A| \geq \kappa$.

Define sets of maps in $\mathscr{C}^{\mathscr{O}_G^{op}}$ by

$$I_{\mathscr{O}_G} := \{G/H \times i\}_{i \in I, H \le G}$$

and

$$J_{\mathscr{O}_G} := \{G/H \times j\}_{j \in J, H \le G}.$$

We then have the following result.

Theorem 3.3 ([Hir] Theorem 11.6.1)**.** *The diagram category $\mathscr{C}^{\mathscr{O}_G^{op}}$ is a cofibrantly generated model category with generating cofibrations $I_{\mathscr{O}_G}$ and generating trivial cofibrations $J_{\mathscr{O}_G}$. The weak equivalences and fibrations are the objectwise weak equivalences and fibrations.*

Remark 3.4. The preceding theorem works equally well if we replace $\mathscr{O}_G^{op}$ by any small category $\mathscr{D}$. The generating cofibrations in this case would be

$$I_{\mathscr{D}} = \{d \times i\}_{d \in \mathscr{D}, i \in I},$$

where for $d \in \mathscr{D}$ and $X \in \mathscr{C}$, the object $d \times X \in \mathscr{C}^{\mathscr{D}}$ is defined by

$$(d \times X)(d') = \mathrm{Hom}(d, d') \times X = \coprod_{\mathrm{Hom}(d,d')} X.$$

Similarly for the generating acyclic cofibrations.

The main tool used in the proof of the above theorem is the following theorem of Dan Kan. Sometimes referred to as the "transfer principle", this is an extremely useful property of cofibrantly generated model categories that allows one to transport model category structures along left adjoints.

Theorem 3.5. *(*[Hir]*, 11.3.2) Let $\mathscr{C}$ be a cofibrantly generated model category with generating cofibrations I and generating acyclic cofibrations J. Let $\mathscr{D}$ be a category which is complete and cocomplete and suppose given an adjoint pair $F : \mathscr{C} \leftrightarrows \mathscr{D} : G$. Then the sets FI and FJ are the generating cofibrations and acyclic cofibrations for a cofibrantly generated model structure on $\mathscr{D}$ if*

1. *the domains of the maps in FI are small with respect to the relative FI-cell complexes, and similarly for FJ, and*
2. *G takes relative FJ-cell complexes to weak equivalences in $\mathscr{C}$.*

The weak equivalences and fibrations are then the the morphisms which become so after application of the functor G.

3.2. G-objects in $\mathscr{C}$

The notion of G-object makes sense in an abritray cofibrantly generated model category, but in order to be able to define weak equivalences of G-objects, we need a fixed points functor for each $H \le G$ (or at least for a family of subgroups).

Definition 3.6. Given a subgroup $H \le G$, an **H-fixed points functor** on $G\mathscr{C}$ is a right adjoint to the "trivial H-space" functor $G/H \times (-) : \mathscr{C} \to H\mathscr{C} \to G\mathscr{C}$.

A moment's reflection shows that for a H-space X, the object X^H is exactly the limit of X, regarded as a functor $H \to \mathscr{C}$. Since we are taking $\mathscr{C}$ to be a model category, it is complete and so fixed point functors exist.

Given such fixed point functors, we define a map of G-objects to be a weak equivalence or fibration if it is so after passage to H fixed points for every $H \le G$.

A cofibration will then be defined to be a map with the left lifting property with respect to acyclic fibrations.

In order to be able to apply the small object argument, we will need to require something of our fixed point functors. We will put a cofibrantly generated model structure on $G\mathscr{C}$ where the generating cofibrations are given by

$$I_G = \{G/H \times i\}_{i \in I, H \leq G},$$

and the generating acyclic cofibrations by

$$J_G = \{G/H \times j\}_{j \in J, H \leq G}.$$

We will need to require that our fixed point functors take I_G-cell complexes to cofibrations in $\mathscr{C}$ and take J_G-cell complexes to acyclic cofibrations in $\mathscr{C}$.

Definition 3.7. A fixed point functor $(-)^H : G\mathscr{C} \to \mathscr{C}$ is said to be a **cellular** fixed point functor if it

(1) preserves pushouts along maps of the form

$$G/K \times X \xrightarrow{1 \times f} G/K \times Y,$$

where X and Y are trivial H-spaces and f is a cofibration in $\mathscr{C}$ and

(2) preserves directed colimits of G-maps which are cofibrations as maps in $\mathscr{C}$.

A fixed point functor automatically takes a map $G/K \times X \xrightarrow{1\times f} G/K \times Y$ to a cofibration if X and Y are trivial G-spaces and f is a cofibration since

$$(G/K \times X)^H \cong \coprod_{G/H \to G/K} X$$

and similarly for Y.

Remark 3.8. The cellularity condition above requires the fixed point functors to preserve certain colimits, but we point out that fixed point functors will usually not preserve *all* colimits. For instance, consider the case $G = \mathbb{Z}/2\mathbb{Z}$ acting on S^1 by rotating through 180°. If $\sigma \in \mathbb{Z}/2\mathbb{Z}$ denotes the nontrivial element, then we have the coequalizer diagram

$$S^1 \underset{\mathrm{id}}{\overset{\sigma}{\rightrightarrows}} S^1 \longrightarrow S^1_{triv}$$

in $\mathbb{Z}/2\mathbb{Z}$-spaces, where $\mathbb{Z}/2\mathbb{Z}$ acts trivially on S^1_{triv}. Taking $\mathbb{Z}/2\mathbb{Z}$-fixed points, however, yields the diagram of spaces

$$\emptyset \rightrightarrows \emptyset \longrightarrow S^1$$

which is certainly not a coequalizer.

Remark 3.9. Note that the conditions for a fixed point functor to be cellular resemble the conditions appearing in Dwyer and Kan's notion of "orbit" [DK], only the latter is an up-to-homotopy version. They can get away with this looser condition since they are working simplicially and all simplicial sets are small with respect to the whole category *sSet*.

The point of requiring fixed point functors to be cellular is that it makes the following true:

Proposition 3.10. *If $G\mathscr{C}$ has cellular fixed point functors, then each $(-)^H : G\mathscr{C} \to \mathscr{C}$ takes I_G-cell complexes to I-cell complexes and J_G-cell complexes to J-cell complexes.*

Once we have fixed point functors for every $H \leq G$, we may define a functor

$$\Phi : G\mathscr{C} \to \mathscr{C}^{\mathscr{O}_G^{op}}$$

as in Section 2. Moreover, Φ has a left adjoint $\Theta : \mathscr{C}^{\mathscr{O}_G^{op}} \to G\mathscr{C}$ given by evaluating at the orbit G/e.

Lemma 3.11. *We have $\Theta \circ \Phi = \mathrm{id}$ and Φ is full and faithful.*

Theorem 3.12. *If $\mathscr{C}$ is a cofibrantly generated model category such that $G\mathscr{C}$ has cellular fixed point functors, then $G\mathscr{C}$ is a cofibrantly generated model category with generating cofibrations*

$$I_G = \{G/H \times i\}_{i \in I, H \leq G},$$

generating acyclic cofibrations

$$J_G = \{G/H \times j\}_{j \in J, H \leq G},$$

and weak equivalences as described above.

Proof. Again, the weak equivalences and fibrations are the maps which are so after passage to H-fixed points for every $H \leq G$, and the cofibrations are the maps having the left lifting property (LLP) with respect to the acyclic fibrations.

Note that an adjointness argument shows that the fibrations are the maps with the RLP with respect to J_G (these are called J-injectives). Similarly the acyclic fibrations are the maps with the RLP with respect to I_G.

Many of the model category axioms for $G\mathscr{C}$ may be deduced from those for $\mathscr{C}$: completeness and cocompleteness, the 2-out-of-3 property for the weak equivalences, and the retract axiom for weak equivalences and fibrations. The retract axiom for cofibrations follows since they are defined by a left lifting property. Before proving the lifting axiom for acyclic cofibrations, we will prove the factorization axioms.

First we show that I_G and J_G permit the small object argument. Suppose that $G/H \times S_i$ is a domain of I_G and let $A \to X$ be a relative I_G-cell complex. By adjointness, the result will follow from smallness of S_i with respect to I if we know that $A^H \to X^H$ is an I-cell complex. But this is Prop. 3.10. The same argument works for J_G.

Let $f : X \to Y$ be an arbitrary G-map. The small object argument then gives us a factorization of f as $X \xrightarrow{i} Z \xrightarrow{p} Y$ where i is a relative I_G-cell complex and therefore a cofibration and where p is an I_G-injective and therefore an acyclic fibration.

For the other factorization axiom, let $f : X \to Y$ be a G-map once again. The small object argument gives us a factorization $X \xrightarrow{j} Z \xrightarrow{q} Y$, where j is a relative J_G-cell complex and q is a J_G injective and therefore a fibration. Note that a relative J_G-cell complex satisfies the LLP with respect to all fibrations; in particular, it satisfies the LLP with respect to acyclic fibrations and so is a cofibraiton. It remains to show that j is a weak equivalence, but this follows from Prop. 3.10.

The remaining lifting axiom follows from the folllowing lemma since J_G-cell complexes satisfy the LLP with respect to fibrations.

Lemma 3.13. *An acyclic cofibration is a retract of a relative J_G-cell complex.*

Proof. Let $A \xrightarrow{i} B$ be an acyclic cofibration. By the factorization axiom proved above, we may factor i as $A \xrightarrow{j} C \xrightarrow{q} B$ where j is a relative J_G-cell complex and q is a fibration. By Prop. 3.10, j is a weak equivalence; the 2-out-of-3 axiom forces q to be an acyclic fibration. But then we get a lift in the diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & C \arrow[d, "q"] \\ B \arrow[r, equal] \arrow[ur, dashed, "\lambda"] & B \end{tikzcd}$$

This shows that i is a retract of j. □

■

Remark 3.14. The short proof from ([MM], Theorem III.1.8) works just as well here.

Proposition 3.15. *With the same assumptions as Theorem 3.12, the adjoint pairs*

$$G/e \times (-) : \mathscr{C} \rightleftarrows G\mathscr{C} : (-)^e$$

and

$$\Theta : \mathscr{C}^{\mathscr{O}_G^{op}} \rightleftarrows G\mathscr{C} : \Phi$$

are Quillen pairs. Moreover, the latter pair is a pair of Quillen equivalences.

Proof. The functor $(-)^e$ preserves weak equivalences and fibrations by definition of the model structure on $G\mathscr{C}$. Similarly, the functor Φ preserves fibrations and weak equivalences.

Recall that (Θ, Φ) is a pair of Quillen equivalences if for every cofibrant $X \in \mathscr{C}^{\mathscr{O}_G^{op}}$ and fibrant $Y \in G\mathscr{C}$ then a morphism

$$f : \Theta(X) \to Y$$

is a weak equivalence if and only if the corresponding map

$$g : X \to \Phi(Y)$$

is a weak equivalence. To show this, we will need the following:

Lemma 3.16. *If $X \in \mathscr{C}^{\mathscr{O}_G^{op}}$ is cofibrant, then the unit of the adjunction*

$$\eta : X \to \Phi\Theta(X)$$

is an isomorphism.

Proof. Recall that the cofibrant objects are the retracts of $I_{\mathscr{O}_G}$-cell complexes. The functor Θ preserves colimits and takes the set $I_{\mathscr{O}_G}$ to the set I_G. Combining this with Prop. 3.10, we have that η is an isomorphism on cell complexes. We are then done since isomorphisms are closed under retracts. □

Given $f : \Theta(X) \to Y$, the morphism $g^H : X(G/H) \to Y^H$ can be factored as

$$X(G/H) \xrightarrow[\cong]{\eta_{G/H}} (X(G/e))^H \xrightarrow{f^H} Y^H.$$

The 2-out-of-3 axiom then shows that f is a weak equivalence if and only if g is. Note that we have not used that Y is fibrant. ■

4. Examples

4.1. Spaces

The case where $\mathscr{C}$ is the category of topological spaces is discussed in [MM], section III.1. That the fixed point functors are cellular is Lemma III.1.6.

4.2. Simplicial Sets

We claim that fixed point functors for G-simplicial sets (=simplicial G-sets) are cellular. To see that they preserve the appropriate pushouts, it suffices to do so for m-simplices for all m; then this turns into an easy question in the category of sets. Similarly for the preservation of colimits of monomorphisms. Thus Theorem 3.12 applies to give a model structure on $GsSet$.

4.3. Simplicial Presheaves

Let $\mathscr{D}$ be a small category. As in Remark 3.4, the diagram category $\mathscr{C}^{\mathscr{D}^{op}}$ becomes a cofibrantly generated model category. Note that the category of G-objects in $\mathscr{C}^{\mathscr{D}^{op}}$ is canonically isomorphic to the category $(G\mathscr{C})^{\mathscr{D}^{op}}$. Specializing to $\mathscr{C} = sSet$, we saw in Example 4.2 that $GsSet$ has cellular fixed point functors, so that $GsSet$ becomes a cofibrantly generated model category. It follows that $G(sSet^{\mathscr{D}^{op}}) \cong (GsSet)^{\mathscr{D}^{op}}$ becomes a cofibrantly generated model category.

Alternatively, we can show directly that $G(sSet^{\mathscr{D}^{op}})$ has cellular fixed point functors. Recall first that every cofibration in $sSet^{\mathscr{D}^{op}}$ is in particular an objectwise cofibration (=monomorphism). Since colimits are computed objectwise, it thus suffices to show that the fixed point functors are objectwise cellular. This follows from Example 4.2.

4.4. "Presented Homotopy Theories"

There is a slightly more restrictive kind of model category, known as a cellular model category (cf. [Hir], 12.1). These include $sSet$ and Top, as well as diagram categories such as $sSet^{\mathscr{D}}$ and $Top^{\mathscr{D}}$. For our purposes, the interest in restricting to cellular model categories is that left proper cellular model categories admit left Bousfield localizations with respect to sets of maps ([Hir], Theorem 4.1.1).

In [D1], Dugger considers model categories obtained by first forming a "free" homotopy theory and then imposing certain relations. More precisely, starting with a small category $\mathscr{D}$, the diagram category $sSet^{\mathscr{D}^{op}}$ with the model structure of Remark 3.4 is the "universal" model category built from $\mathscr{D}$. Since $sSet^{\mathscr{D}^{op}}$ is the category of simplicial presheaves on $\mathscr{D}$, we will write $sPre(\mathscr{D})$ for $sSet^{\mathscr{D}^{op}}$. This is a left proper cellular model category, essentially inheriting these properties from $sSet$. Thus given any set A of maps in $sPre(\mathscr{D})$, there exists a left Bousfield localization $L_A sPre(\mathscr{D})$ (this is simply a new model structure on the same underlying category), and this model structure on $sPre(\mathscr{D})$ is again left proper and cellular.

One may think of this as giving a "presentation" for the resulting homotopy category (Dugger speaks of presentations of model categories rather than of homotopy categories); the category $\mathscr{D}$ provides the generators while A gives the relations. Many model categories are Quillen equivalent to "presented" model categories; in particular, Dugger has shown ([D2]) that J. Smith's *combinatorial* model categories have presentations.

Now let $L_A sPre(\mathscr{D})$ be a presentation for a homotopy theory. As we have seen in Example 4.3, $GsPre(\mathscr{D})$ has cellular fixed point functors. But it then follows that $G\big(L_A sPre(\mathscr{D})\big)$ also has cellular fixed point functors since (1) $L_A sPre(\mathscr{D})$ and $sPre(\mathscr{D})$ have the same underlying category and (2) the cofibrations of $L_A sPre(\mathscr{D})$ are exactly the cofibrations of $sPre(\mathscr{D})$. Thus Theorem 3.12 applies to give us a model structure on $G\big(L_A sPre(\mathscr{D})\big)$.

4.5. Homotopy Theory of Schemes

We begin by recalling the Morel-Voevodsky unstable motivic homotopy category ([MV]). Fix a perfect field k and consider the category $\mathrm{Sm_k}$ of smooth schemes of finite type over k. There is a Grothendieck topology, called the Nisnevich topology, on this category. It is finer than the Zariski topology but coarser than the étale; in fact it is fine enough to allow for a "homotopy purity" theorem but coarse enough so that algebraic K-theory satisfies descent. The details need not concern us. One then considers the category $sSh_{Nis}(\mathrm{Sm_k})$ of simplicial Nisnevich sheaves on $\mathrm{Sm_k}$. This has a model structure, due to Joyal, in which the weak equivalences are the stalkwise weak equivalences and the cofibrations are the monomorphisms. One then performs a left Bousfield localization at the set of maps $X \times \mathbb{A}^1 \to X$, and the resulting homotopy category is the unstable motivic homotopy category $Ho(k)$.

Jardine has shown ([J]) that there is a model structure on the category $sPre(\mathrm{Sm_k})$ of simplicial *pre*sheaves where the cofibrations are the monomorphisms and the weak equivalences are the maps which induce isomorphisms on sheaves of homotopy groups. Sheafification then induces a Quillen equivalence

$$a_{Nis} : sPre(\mathrm{Sm_k}) \rightleftarrows sShv_{Nis}(\mathrm{Sm_k}) : i$$

(the right adjoint is the inclusion), and this Quillen equivalence descends to $\mathbb{A}^1$-localizations.

Finally, as Dugger explains in [D1], one can give a presentation as follows. First consider $sPre(\mathrm{Sm_k})$ with the objectwise weak equivalences and fibrations (the model structure from Remark 3.4). One then localizes at the Nisnevich *hypercovers* $U_* \to X$ as well as at the projections $X \times \mathbb{A}^1 \to X$. The identity functor on $sPre(\mathrm{Sm_k})$ then is a left adjoint of a Quillen equivalence from this model structure to Jardine's model structure (this is Theorem 6.2 of [DHI]).

Thus the Morel-Voevodsky homotopy category fits into Example 4.4 and we get a model structure on G-motivic spaces.

5. Coarse G-structure

The model structure of Theorem 3.12 is sometimes called the *fine* model structure on G-spaces. There is another interesting model structure, called the *coarse* model structure, in which one only considers the trivial subgroup $1 \leq G$.

Theorem 5.1 (Coarse Model Structure)**.** *There is a cofibrantly generated model structure on $G\mathscr{C}$, where the weak equivalences and fibrations are the maps which are so as maps in $\mathscr{C}$. The generating cofibrations are given by $G \times I$ and the generating acyclic cofibrations are given by $G \times J$.*

One can prove this either by repeating the proof of Theorem 3.12, using only the trivial subgroup instead of all of the groups, or by noting that this is another special case of 3.4 with $\mathscr{D}$ the group G regarded as a one object category with morphisms given by G.

We now mention some trivial consequences of the existence of these model structures.

Proposition 5.2. *The adjunction* $\mathrm{id} : G\mathscr{C} \rightleftarrows G\mathscr{C} : \mathrm{id}$ *is a Quillen adjunction from the coarse model structure to the fine model structure.*

Proposition 5.3. *The cofibrant objects in the coarse model structure have free G-action. In particular, a cofibrant replacement $\tilde{X} \to X$ is a G-object with free G-action which is weakly equivalent to X in $\mathscr{C}$.*

Example 5.4. Thus we see that in spaces, $EG \to *$ is a cofibrant replacement for $*$ in the coarse model structure.

Example 5.5. The quotient functor $Q : G\mathscr{C} \to \mathscr{C}$ given by $Q(X) = X/G$ has right adjoint the trivial G-space functor $t : \mathscr{C} \to G\mathscr{C}$, and it is clear that t preserves fibrations and weak equivalences, so that this becomes a Quillen pair. The left derived functor of Q is given by $\mathbf{L}Q(X) = EG \times_G X$, the Borel construction.

Example 5.6. Similarly, for any G-space X, the functor $\mathrm{Map}_G(-, X) : G\mathscr{C}^{op} \to \mathscr{C}$ has a left adjoint given by $\mathrm{Map}(-, X)^{op} : \mathscr{C} \to G\mathscr{C}^{op}$. The category $G\mathscr{C}^{op}$ inherits the opposite model structure from $G\mathscr{C}$, and one gets $\mathbf{R}\,\mathrm{Map}_G(*, X) = \mathrm{Map}_G(EG, X)$.

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