Math 551 - Topology I
Homework 1
Fall 2013

1. Show that the max metric on $\mathbb{R}^2$ is a metric. Recall that the max metric is defined by
$$d(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

2. (a) Let X be any set. Define a metric on X by
$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases}$$
Show that this is indeed a metric. This is called the discrete metric.
   (b) Which sets are open in the discrete metric?
   (c) Suppose that X is a discrete metric space and that Y is any metric space. Show that any function $f : X \longrightarrow Y$ is automatically continuous.
   (d) ($\star$) Suppose that Y is a discrete metric space. Show that the only continuous functions $\mathbb{R} \longrightarrow Y$ are the constant functions.

3. Show that $U \subseteq \mathbb{R}^2$ is open in the max metric if and only if it is open in the standard metric.

4. Show that, for a function $f : X \longrightarrow Y$ between metric spaces, the following are equivalent.
   (2) for every $x \in X$ and for every $\varepsilon > 0$, there is a $\delta > 0$ such that
$$B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x)))$$
   (3) For every $y \in Y$ and $\epsilon > 0$ and $x \in X$, if $f(x) \in B_\epsilon(y)$, then there exists a $\delta > 0$ such that
$$B_\delta(x) \subseteq f^{-1}(B_\epsilon(y))$$

5. Let $f : X \longrightarrow Y$ be a function between metric spaces. Show that f is continuous if and only if
   For every convergent sequence $(x_n) \to x$ in X, the sequence $(f(x_n))$ converges to $f(x)$ in Y.

6. Show that the intersection of two open sets in a metric space is always open.