1. Show that the max metric on $\mathbb{R}^2$ is a metric. Recall that the max metric is defined by

$$d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$ 

2. (a) Let $X$ be any set. Define a metric on $X$ by

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases}$$

Show that this is indeed a metric. This is called the discrete metric.

(b) Which sets are open in the discrete metric?

(c) Suppose that $X$ is a discrete metric space and that $Y$ is any metric space. Show that any function $f : X \to Y$ is automatically continuous.

(d) $(\star)$ Suppose that $Y$ is a discrete metric space. Show that the only continuous functions $\mathbb{R} \to Y$ are the constant functions.

3. Show that $U \subseteq \mathbb{R}^2$ is open in the max metric if and only if it is open in the standard metric.

4. Show that, for a function $f : X \to Y$ between metric spaces, the following are equivalent.

(2) for every $x \in X$ and for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$B_{\delta}(x) \subseteq f^{-1}(B_{\varepsilon}(f(x)))$$

(3) For every $y \in Y$ and $\varepsilon > 0$ and $x \in X$, if $f(x) \in B_{\varepsilon}(y)$, then there exists a $\delta > 0$ such that

$$B_{\delta}(x) \subseteq f^{-1}(B_{\varepsilon}(y))$$

5. Let $f : X \to Y$ be a function between metric spaces. Show that $f$ is continuous if and only if

For every convergent sequence $(x_n) \to x$ in $X$, the sequence $(f(x_n))$ converges to $f(x)$ in $Y$.

6. Show that the intersection of two open sets in a metric space is always open.