Math 551 - Topology I
Homework 8
Fall 2013

1. (Metric completion) Let X be a metric space. Let C_X be the set of Cauchy sequences in X. Define a relation on C_X by

$$(x_n) \sim (y_n) \quad \text{if} \quad \lim_n d_X(x_n, y_n) = 0.$$

Define $X^* = (C_X/\sim)$. Then $d\Big((x_n), (y_n)\Big) = \lim_n d_X(x_n, y_n)$ defines a metric on X^*. Then the map

$$\begin{aligned} \iota : X &\longrightarrow X^* \\ x &\mapsto (x, x, \dots) \end{aligned}$$

is an isometric embedding.

   (a) Show that $\iota(X)$ is dense in X^*.

   (b) Show that if A is a dense subset of a metric space Z and every Cauchy sequence in A converges in Z, then Z is complete. Conclude that X^* is complete.

   (c) Show that X is totally bounded if and only if X^* is compact.

2. A space satisfying the conclusion of the Baire Category Theorem is called a **Baire space**.

   (a) Show that the irrationals are a Baire space.

   (b) Show that $\mathbb{Q}$ is not a Baire space.

3. A subset $A \subseteq X$ is said to be **nowhere dense** in X if $\text{Int}(\overline{A}) = \emptyset$.

   (a) Show that $A \subseteq X$ is closed and nowhere dense if and only if $A = \partial U$ for some open $U \subseteq X$.

   (b) Show that a nonempty Baire space X *cannot* be expressed as a countable union of nowhere dense sets.