CLASS NOTES
MATH 551 (FALL 2013)

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1. Wed, Aug. 28

Topology is the study of shapes. (The Greek meaning of the word is the study of places.) What kind of shapes? Many are familiar objects: a circle or triangle or square. Going up in dimension, we might want to study a sphere or box or a torus.

In fact, all of these arise as **metric spaces**, but topology is quite a bit more general. For starters, a circle of radius 1 is the same as a circle of radius 123978632 from the eyes of topology. We will also see that there are many interesting spaces that can be obtained by modifying familiar metric spaces, but the resulting spaces cannot always be given a nice metric.

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As we said, many examples that we care about are metric spaces, so we'll start by reviewing the theory of metric spaces.

Definition 1.1. A **metric space** is a pair (X, d), where X is a set and $d : X \times X \longrightarrow \mathbb{R}$ is a function (called a "metric") satisfying the following three properties:

(1) (Symmetry) $d(x, y) = d(y, x)$ for all $x, y \in X$
(2) (Positive-definite) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$
(3) (Triangle Inequality) $d(x, y) + d(y, z) \geq d(x, z)$ for all x, y, z in X.

Example 1.2. (1) $\mathbb{R}$ is a metric space, with $d(x, y) = |x - y|$.
(2) $\mathbb{R}^2$ is a metric space, with $d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. This is called the standard, or Euclidean metric, on $\mathbb{R}^2$.
(3) $\mathbb{R}^n$ similarly has a Euclidean metric, defined by
$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$
(4) $\mathbb{R}^2$, with $d(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.
(5) $\mathbb{R}^2$, with $d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2|$.

Given a point x in a metric space X, we can consider those points "near to x".

Definition 1.3. Let (X, d) be a metric space and let $x \in X$. We define the (open) **ball of radius** r **around** x to be
$$B_r(x) = \{y \in X \mid d(x, y) < r\}.$$

Example 1.4. (1) In $\mathbb{R}$, with the usual metric, we have $B_r(x) = (x - r, x + r)$.
(2) In $\mathbb{R}^2$, with the standard metric, we have $B_r(\mathbf{x})$ is a disc of radius r, centered at $\mathbf{x}$.

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(3) In $\mathbb{R}^n$, with the standard metric, we have $B_r(\mathbf{x})$ is an n-dimensional ball of radius r, centered at $\mathbf{x}$.
(4) In $\mathbb{R}^2$, with the max metric, $B_r(\mathbf{x})$ takes the form of a square, with sides of length $2r$, centered at $\mathbf{x}$.
(5) In $\mathbb{R}^2$, with the "taxicab" metric, $B_r(\mathbf{x})$ is a diamond, with sides of length $r\sqrt{2}$, centered at $\mathbf{x}$.

In the definition of a metric space, we had a metric function $X \times X \longrightarrow \mathbb{R}$. Let's review: what is the set $X \times X$? More generally, what is $X \times Y$, when X and Y are sets. We know this as the set of ordered pairs

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

This is the usual definition of the **cartesian product** of two sets. One of the points of emphasis in this class will be not just objects or constructions but rather maps into/out of objects. With that in mind, given the cartesian product $X \times Y$, can we say anything about maps into or out of $X \times Y$?

The first thing to note is that there are two "natural" maps out of the product; namely, the projections. These are

$$p_X : X \times Y \longrightarrow X, \qquad p_X(x, y) = x$$

and

$$p_Y : X \times Y \longrightarrow Y \qquad p_Y(x, y) = y.$$

Now let's consider functions *into* $X \times Y$ from other, arbitrary, sets. Suppose that Z is a set. How would one specify a function $f : Z \longrightarrow X \times Y$? For each $z \in Z$, we would need to give a point $f(z) \in X \times Y$. This point can be described by listing its X and Y coordinates. Given that the projection p_X takes a point in the product and picks out its X-coordinate, it follows that the function f_X defined as the composition

$$Z \xrightarrow{f} X \times Y \xrightarrow{p_X} X$$

is the function of X-coordinates of the function f. We similarly get a function f_Y by using p_Y instead.

And the main point of this is that the function f contains the same information as the pair of functions f_X and f_Y.

Proposition 1.5. *(Universal property of the cartesian product) Let X, Y, and Z be any sets. Suppose given functions $f_X : Z \longrightarrow X$ and $f_Y : Z \longrightarrow Y$. Then there exists a* **unique** *function $f : Z \longrightarrow X \times Y$ such that*

$$f_X = p_X \circ f, \qquad \textit{and} \qquad f_Y = p_X \circ f.$$

$$\begin{tikzcd}
& & X \\
Z \arrow[r, dashed, "f"] \arrow[urr, bend left, "f_X"] \arrow[drr, bend right, "f_Y"'] & X \times Y \arrow[ur, "p_X"'] \arrow[dr, "p_Y"] & \\
& & Y
\end{tikzcd}$$

Furthermore, it turns out that the above property uniquely characterizes the cartesian product $X \times Y$, up to bijection. We called this a "Proposition", but there is nothing difficult about this, once you understand the statement. The major advance at this point is simply the reframing of a familiar concept. We will see later in the course why this is useful.

As we already said, we will promote the viewpoint that it is not just objects that are important, but also maps. We have introduced the concept of a metric space, so we should then ask "What are maps between metric spaces"?

The strictest answer is what is known as an **isometry**: a function $f : X \longrightarrow Y$ such that $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$ for all pairs of points x_1 and x_2 in X. This is a perfectly fine answer in many regards, but for our purposes, it will be too restrictive. For instance, what are all isometries $\mathbb{R} \longrightarrow \mathbb{R}$?

We will prefer to study the more general class of **continuous** functions.

2. Fri, Aug. 30

Definition 2.1. A function $f : X \longrightarrow Y$ between metric spaces is **continuous** if for every $x \in X$ and for every $\varepsilon > 0$, there is a $\delta > 0$ such that whenever $x' \in B_\delta(x)$, then $f(x') \in B_\varepsilon(f(x))$.

This is the standard definition, taken straight from Calc I and written in the language of metric spaces. However, it is not always the most convenient formulation.

Proposition 2.2. *Let $f : X \longrightarrow Y$ be a function between metric spaces. The following are equivalent:*

(1) *f is continuous*
(2) *for every $x \in X$ and for every $\varepsilon > 0$, there is a $\delta > 0$ such that*
$$B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x)))$$
(3) *For every $y \in Y$ and $\epsilon > 0$ and $x \in X$, if $f(x) \in B_\epsilon(y)$, then there exists a $\delta > 0$ such that*
$$B_\delta(x) \subseteq f^{-1}(B_\epsilon(y))$$
(4) *For every $y \in Y$ and $\epsilon > 0$ and $x \in X$, if $x \in f^{-1}(B_\epsilon(y))$, then there exists a $\delta > 0$ such that*
$$B_\delta(x) \subseteq f^{-1}(B_\epsilon(y))$$

The property that $f^{-1}(B_\epsilon(y))$ satisfies in condition (4) is important, and we give it a name:

Definition 2.3. Let $U \subseteq X$ be a subset. We say that U is **open** in X if whenever $x \in U$, then there exists a $\delta > 0$ such that $B_\delta(x) \subseteq U$.

With this language at hand, we can restate condition (4) above as

(4′) For every $y \in Y$ and $\epsilon > 0$, $f^{-1}(B_\epsilon(y))$ is open in X.

The language suggests that an open ball should count as an open set, and this is indeed true.

Proposition 2.4. *Let $c \in X$ and $\epsilon > 0$. Then $B_\epsilon(c)$ is open in X.*

Proof. Suppose $x \in B_\epsilon(c)$. This means that $d(x, c) < \epsilon$. Write d for this distance. Let
$$\delta = \epsilon - d.$$
We claim that this is the desired δ. For suppose that $u \in B_\delta(x)$. Then
$$d(u, c) \leq d(u, x) + d(x, c) < \delta + d = \epsilon.$$
(Draw a picture!) ■

Ok, so the notion of open set is closely related to that of open ball: every open ball is an open set, and every open set is required to contain a number of these open balls. Even better, we have the following result:

Proposition 2.5. *A subset $U \subseteq X$ is open if and only if it can be expressed as a union of open balls.*

Proof. Suppose U is open, and let $x \in U$. By definition, there exists $\delta_x > 0$ with $B_{\delta_x}(x) \subseteq U$. Since this is true for every $x \in U$, we have

$$\bigcup_{x \in U} B_{\delta_x}(x) \subseteq U.$$

But every $x \in U$ is contained in the union, so clearly U must also be contained in the union. It follows that

$$\bigcup_{x \in U} B_{\delta_x}(x) = U.$$

Now suppose, on the other hand, that $U = \bigcup_\alpha B_{\delta_\alpha}(x_\alpha)$. We wish to show that U is open. Well, suppose $u \in U$. Since U is expressed as a union, this implies that $u \in B_{\delta_\alpha}(x_\alpha)$ for some α. This ball is contained in U by the definition of U, so we are done. ∎

Corollary 2.6. *Any union of open subsets of X is open.*

With this description of open sets in hand, we give what is often the most useful characterization of continuous maps.

Proposition 2.7. *Let $f : X \longrightarrow Y$ be a function between metric spaces. The following are equivalent:*

(1) *f is continuous*
(5) *For every open subset $V \subseteq Y$, the preimage $f^{-1}(V)$ is open in X.*

Proof. It is clear that (5) implies (4′), which is equivalent to (1) by Prop 2.2. Now assume (1), or, equivalently, (4′). Let $V \subseteq Y$ be open. By the previous result, V is a union of balls, and by (4′) we know that the preimage of each ball is open. Using Corollary 2.6, it follows that $f^{-1}(V)$ is open. ∎

For example, let's show that the translation map $t : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $t(x) = x+1$ is continuous, but that

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \qquad f(x) = \begin{cases} x & x < 0 \\ x+2 & x \geq 0 \end{cases}$$

is not continuous. As we already said, a ball in $\mathbb{R}$ is an open interval, and

$$t^{-1}(a,b) = (a-1, b-1)$$

is certainly open. On the other hand, $(1,3)$ is open but $f^{-1}(1,3) = [0,1)$ is not (since it contains 0 but no ball centered at 0).

In calculus, we are also used to thinking of continuity in terms of convergence of sequences. Recall that a sequence (x_n) in X **converges** to x if for every $\epsilon > 0$ there exists N such that for all $n > N$, we have $x_n \in B_\epsilon(x)$. We say that a "tail" of the sequence is contained in the ball around x.

Proposition 2.8. *Let $f : X \longrightarrow Y$ be a function between metric spaces. The following are equivalent:*

(1) *f is continuous*
(6) *For every convergent sequence $(x_n) \to x$ in X, the sequence $(f(x_n))$ converges to $f(x)$ in Y.*

Proof. Suppose that f is continuous and assume $x_n \to x$. We want to show that $f(x_n) \to f(x)$. So let $f(x) \in V \subseteq Y$ be open. By assumption, $f^{-1}(V)$ is open, and $x \in f^{-1}(V)$. By the definition of convergence, it follows that a tail of this sequence is in $f^{-1}(V)$. Now apply f, and we find that the image of that tail (in other words, a tail of the image sequence) is contained in V.

We will deal with the other implication next time. ∎