Last time, we introduced the idea of a completely regular space, and we saw that this allowed us to build a universal compactification (the Stone-Čech compactification).

**Corollary 32.1.** $X$ is completely regular if and only if it is homeomorphic to a subspace of $[0, 1]^J$ for some set $J$.

This has consequences for metrizability of a space. Consider first the case that the index set $J$ is countable.

**Proposition 32.2.** Let $Y$ be a metric space, and let $d : Y \times Y \to \mathbb{R}$ be the associated truncated metric. Then the formula

$$D(y, z) = \sup \left\{ \frac{d(y_n, z_n)}{n} \right\}$$

defines a metric on $Y^\infty$, and the induced topology is the product topology.

**Proof.** We leave as an exercise the verification that this is a metric. We check the statement about the topology. For each $n$, let $p_n : Y^\infty \to Y$ be evaluation in the $n$th place. This is continuous, as given $y \in Y^\infty$ and $\varepsilon > 0$, we let $\delta = \varepsilon/n$. Then if $D(y, z) < \delta$, it follows that

$$d(y_n, z_n) = n \frac{d(y_n, z_n)}{n} \leq nD(y, z) < n\delta = \varepsilon.$$

By the universal property of the product, we get a continuous bijection $p : Y^\infty \to \prod_{i=1}^{\infty} Y$.

It remains to show that $p$ is open. Thus let $B \subseteq Y^\infty$ be an open ball, and let $y \in p(B) = B$. We want to find a basis element $U$ in the product topology with $y \in U \subseteq B$. For convenience, we replace $B$ by $B_\varepsilon(y)$ for small enough $\varepsilon$. Take $N$ large such that $1/N < \varepsilon$. Then define

$$U = \bigcap_{i=1}^{N} p_{i}^{-1}(B_\varepsilon(y_i)).$$

Let $z \in Y^\infty$. Recall that we have truncated our metric on $Y$ at 1. Thus if $n > N$, we have that $\bar{d}(y_n, z_n)/n \leq 1/n \leq 1/N < \varepsilon$. It follows that for any $z \in U$, we have $z \in B_\varepsilon(x)$ as desired.

On the other hand, if $J$ is uncountable, then $[0, 1]^J$ need not be metric, as the following example shows.

**Example 32.3.** The sequence lemma fails in $\mathbb{R}^\mathbb{R}$. Let $A \subseteq \mathbb{R}^\mathbb{R}$ be the subset consisting of functions that zero at all but finitely many points. Let $g$ be the constant function at 1. Then $g \in \overline{A}$, since if $U = \bigcap_{x_1, \ldots, x_k} p_{x_i}^{-1}(a_i, b_i)$ is a neighborhood of $g$, then the function

$$f(x) = \begin{cases} 1 & x \in \{x_1, \ldots, x_k\} \\ 0 & \text{else} \end{cases}$$

is in $U \cap A$. But no sequence in $A$ can converge to $g$ (recall that convergence in the product topology means pointwise convergence). For suppose $f_n$ is a sequence in $A$. For each $n$, let $Z_n = \text{supp}(f_n)$ (the support is the set where $f_n$ is nonzero). Then the set

$$Z = \bigcup_{n} Z_n$$

is countable, and on the complement of $Z$, all $f_n$'s are zero. So it follows that the same must be true for any limit of $f_n$. Thus the $f_n$ cannot converge to $g$. 52
This finally leads to a characterization of those topological spaces which come from metric spaces.

**Theorem 32.4** (Munkres, Theorem 32.1). *If X is second countable and regular, then it is normal.*

**Theorem 32.5.** *If X is regular and second countable, then it is metrizable.*

Proof. Since X is completely regular, we can embed X as above inside a cube \([0,1]^J\) for some \(J\). Above, we took \(J\) to be the collection of all functions \(X \rightarrow [0,1]\).

To get a countable indexing set \(J\), start with a countable basis \(B = \{B_n\}\) for \(X\). For each pair of indices \(n, m\) for which \(B_n \subset B_m\), the Urysohn lemma gives us a function \(g_{n,m}\) vanishing on \(B_n\) and equal to 1 outside \(B_m\). We take \(J = \{g_{n,m}\}\). Going back to the proof of the Stone-Čech compactification, we needed, for any \(x_0 \in X\) and \(x_0 \in U\), to be able to find a function vanishing at \(x_0\) but equal to 1 outside of \(U\).

Take a basis element \(B_m\) satisfying \(x_0 \in B_m \subset U\). Since \(X\) is regular, we can find an open set \(V\) with \(x_0 \in V \subset \overline{V} \subset B_m\). Find a \(B_n\) inside of \(V\), and we are now done: namely, the function \(g_{n,m}\) is what we were after.

33. Wed, Nov. 13

**Midterm! part deux.**

34. Fri, Nov. 15

Recall that a space is paracompact if every open cover has a locally finite open refinement that is a cover. We saw that any second countable Hausdorff space is paracompact.

**Theorem 34.1** (Munkres, Theorem 41.4). *If X is metric, then it is paracompact.*

As we will see, paracompactness will allow us to build functions. As a first step, we show that this implies normality. First, we need a lemma.

**Lemma 34.2.** *If \(\{A\}\) is a locally finite collection of subsets of \(X\), then\[
\bigcup A = \bigcup \overline{A}.
\]

Proof. We have already shown before that the inclusion \(\supset\) holds generally. The other implication follows from the neighborhood criterion for the closure. Let \(x \in \bigcup A\). Then we can find a neighborhood \(U\) of \(x\) meeting only \(A_1, \ldots, A_n\). Then \(x \in \bigcup_{i=1}^n A_i\) since else there would be a neighborhood \(V\) of \(x\) meeting the \(A_i\)'s. Then \(U \cap V\) would be a neighborhood missing \(\bigcup A\). But \(\bigcup_{i=1}^n A_i = \bigcup_{i=1}^n \overline{A_i}\), so we are done.

**Theorem 34.3** (Lee, Theorem 4.81). *If X is paracompact and Hausdorff, then it is normal.*

Proof. We first use the Hausdorff assumption to show that \(X\) is regular. A similar argument can then be made, using regularity, to show normality.

Thus let \(A\) be closed and \(b \notin A\). We wish to find disjoint open sets \(A \subset U\) and \(b \in V\). For every \(a \in A\), we can find disjoint open neighborhoods \(U_a\) of \(a\) and \(V_a\) of \(b\). Then \(\{U_a\} \cup \{X \setminus A\}\) is an open cover, so there is a locally finite subcover \(\mathcal{U}\). Take \(W \subset \mathcal{U}\) to be the \(W \subset \mathcal{U}\) such that \(W \subset U_a\) for some \(a\). Then \(\mathcal{W}\) is still locally finite.

We take \(U = \bigcup_{W \in \mathcal{W}} W\) and \(V = X \setminus \overline{U}\). We know \(b \in V\) since \(\overline{U} = \bigcup W\), and \(b \notin \overline{W}\) since \(W \subset U_a\) and \(b\) has a neighborhood \((V_a)\) disjoint from \(U_a\).

Recall that the **support** of a continuous function \(f: X \rightarrow \mathbb{R}\) is supp\((f) = f^{-1}(\mathbb{R} \setminus \{0\})\).

**Definition 34.4.** Let \(\mathcal{U} = \{U_a\}\) be a cover of \(X\). A **partition of unity** subordinate to \(\mathcal{U}\) is a collection \(\varphi_a : X \rightarrow [0,1]\) of continuous functions such that
(1) $\text{supp}(\varphi_\alpha) \subseteq U_\alpha$
(2) the collection $\text{supp}(\varphi_\alpha)$ is locally finite
(3) we have $\sum_\alpha \varphi_\alpha = 1$. Note that, when evaluated at some $x \in X$, this sum is always finite by the local finite assumption (2).

**Theorem 34.5.** Let $X$ be paracompact Hausdorff, and let $\mathcal{U} = \{U_\alpha\}$ be an open cover. Then there exists a partition of unity subordinate to $\mathcal{U}$.

**Lemma 34.6** (Lee, 4.84). There exists a locally finite refinement $\{V_\alpha\}$ of $\{U_\alpha\}$ with $\overline{V_\alpha} \subseteq U_\alpha$.

**Proof of Theorem.** We apply the lemma twice to get locally finite covers $\{V_\alpha\}$ and $\{W_\alpha\}$ with $\overline{W_\alpha} \subseteq V_\alpha \subseteq \overline{V_\alpha} \subseteq U_\alpha$. For each $\alpha$, we use Urysohn’s lemma to get $f_\alpha : X \rightarrow [0, 1]$ with $f_\alpha \equiv 1$ on $\overline{W_\alpha}$ and $\text{supp}(f_\alpha) \subseteq \overline{V_\alpha} \subseteq U_\alpha$. Since $\{V_\alpha\}$ is locally finite, we can define $f : X \rightarrow [0, 1]$ by $f = \sum_\alpha f_\alpha$. Locally around some $x \in X$, the function $f$ is a finite sum of $f_\alpha$’s, and so is continuous. It only remains to normalize our $f_\alpha$’s. Note that at any $x \in X$, we can find an $\alpha$ for which $x \in W_\alpha$, and so $f(x) \geq f_\alpha(x) = 1$. Thus it makes sense to define $\varphi_\alpha : X \rightarrow [0, 1]$ by

$$\varphi_\alpha(x) = \frac{f_\alpha(x)}{f(x)}.$$  

We have $\text{supp}(\varphi_\alpha) = \text{supp}(f_\alpha)$, and so the $\varphi_\alpha$ give a partition of unity. □

Partitions of unity come up often in the theory of vector bundles. For us, they have an powerful consequence for a very important class of spaces.