Last time, we introduced the idea of a completely regular space, and we saw that this allowed us to build a universal compactification (the Stone-Čech compactification).

Corollary 32.1. *X is completely regular if and only if it is homeomorphic to a subspace of $[0,1]^J$ for some set J.*

This has consequences for *metrizability* of a space. Consider first the case that the index set J is countable.

Proposition 32.2. *Let Y be a metric space, and let $\overline{d} : Y \times Y \longrightarrow \mathbb{R}$ be the associated truncated metric. Then the formula*

$$D(\mathbf{y}, \mathbf{z}) = \sup \left\{ \frac{\overline{d}(y_n, z_n)}{n} \right\}$$

defines a metric on $Y^{\mathbb{N}}$, and the induced topology is the product topology.

Proof. We leave as an exercise the verification that this is a metric. We check the statement about the topology. For each n, let $p_n : Y^{\mathbb{N}} \longrightarrow Y$ be evaluation in the nth place. This is continuous, as given $\mathbf{y} \in Y^{\mathbb{N}}$ and $\epsilon > 0$, we let $\delta = \epsilon/n$. Then if $D(\mathbf{y}, \mathbf{z}) < \delta$, it follows that

$$d(y_n, z_n) = n\frac{d(y_n, z_n)}{n} \leq nD(\mathbf{y}, \mathbf{z}) < n\delta = \epsilon.$$

By the universal property of the product, we get a continuous bijection $p : Y^{\mathbb{N}} \longrightarrow \prod_{\mathbb{N}} Y$.

It remains to show that p is open. Thus let $B \subseteq Y^{\mathbb{N}}$ be an open ball, and let $\mathbf{y} \in p(B) = B$. We want to find a basis element U in the product topology with $\mathbf{y} \in U \subseteq B$. For convenience, we replace B by $B_\epsilon(\mathbf{y})$ for small enough ϵ. Take N large such that $1/N < \epsilon$. Then define

$$U = \bigcap_{i=1}^{N} p_i^{-1}(B_\epsilon(y_i)).$$

Let $\mathbf{z} \in Y^{\mathbb{N}}$. Recall that we have truncated our metric on Y at 1. Thus if $n > N$, we have that $\overline{d}(y_n, z_n)/n \leq 1/n \leq 1/N < \epsilon$. It follows that for any $\mathbf{z} \in U$, we have $\mathbf{z} \in B_\epsilon(\mathbf{x})$ as desired. ∎

On the other hand, if J is uncountable, then $[0,1]^J$ need not be metric, as the following example shows.

Example 32.3. The sequence lemma fails in $\mathbb{R}^{\mathbb{R}}$. Let $A \subseteq \mathbb{R}^{\mathbb{R}}$ be the subset consisting of functions that zero at all but finitely many points. Let g be the constant function at 1. Then $g \in \overline{A}$, since if

$$U = \bigcap_{x_1, \dots, x_k} p_{x_i}^{-1}(a_i, b_i)$$

is a neighborhood of g, then the function

$$f(x) = \begin{cases} 1 & x \in \{x_1, \dots, x_k\} \\ 0 & \text{else} \end{cases}$$

is in $U \cap A$. But no sequence in A can converge to g (recall that convergence in the product topology means pointwise convergence). For suppose f_n is a sequence in A. For each n, let $Z_n = \mathrm{supp}(f_n)$ (the support is the set where f_n is nonzero). Then the set

$$\mathcal{Z} = \bigcup_n Z_n$$

is countable, and on the complement of $\mathcal{Z}$, all f_n's are zero. So it follows that the same must be true for any limit of f_n. Thus the f_n cannot converge to g.

This finally leads to a characterization of those topological spaces which come from metric spaces.

Theorem 32.4 (Munkres, Theorem 32.1)**.** *If X is second countable and regular, then it is normal.*

Theorem 32.5. *If X is regular and second countable, then it is metrizable.*

Proof. Since X is completely regular, we can embed X as above inside a cube $[0,1]^J$ for some J. Above, we took J to be the collection of all functions $X \longrightarrow [0,1]$.

To get a countable indexing set J, start with a countable basis $\mathcal{B} = \{B_n\}$ for X. For each pair of indices n, m for which $\overline{B}_n \subset B_m$, the Urysohn lemma gives us a function $g_{n,m}$ vanishing on $\overline{B}_n$ and equal to 1 outside B_m. We take $J = \{g_{n,m}\}$. Going back to the proof of the Stone-Čech-compactification, we needed, for any $x_0 \in X$ and $x_0 \in U$, to be able to find a function vanishing at x_0 but equal to 1 outside of U.

Take a basis element B_m satisfying $x_0 \in B_m \subset U$. Since X is regular, we can find an open set V with $x_0 \in V \subset \overline{V} \subset B_m$. Find a B_n inside of V, and we are now done: namely, the function $g_{n,m}$ is what we were after. ■

33. Wed, Nov. 13

Midterm! part deux.

34. Fri, Nov. 15

Recall that a space is paracompact if every open cover has a *locally finite open refinement* that is a cover. We saw that any second countable Hausdorff space is paracompact.

Theorem 34.1 (Munkres, Theorem 41.4)**.** *If X is metric, then it is paracompact.*

As we will see, paracompactness will allow us to build functions. As a first step, we show that this implies normality. First, we need a lemma.

Lemma 34.2. *If $\{A\}$ is a locally finite collection of subsets of X, then*

$$\overline{\bigcup A} = \bigcup \overline{A}.$$

Proof. We have already shown before that the inclusion $\supset$ holds generally. The other implication follows from the neighborhood criterion for the closure. Let $x \in \overline{\bigcup A}$. Then we can find a neighborhood U of x meeting only $A_1, \dots, A_n$. Then $x \in \overline{\bigcup_{i=1}^n A_i}$ since else there would be a neighborhood V of x meeting the A_i's. Then $U \cap V$ would be a neighborhood missing $\bigcup A$. But $\overline{\bigcup_{i=1}^n A_i} = \bigcup_{i=1}^n \overline{A_i}$, so we are done. ■

Theorem 34.3 (Lee, Theorem 4.81)**.** *If X is paracompact and Hausdorff, then it is normal.*

Proof. We first use the Hausdorff assumption to show that X is regular. A similar argument can then be made, using regularity, to show normality.

Thus let A be closed and $b \notin A$. We wish to find disjoint open sets $A \subseteq U$ and $b \in V$. For every $a \in A$, we can find disjoint open neighborhoods U_a of a and V_a of b. Then $\{U_a\} \cup \{X \setminus A\}$ is an open cover, so there is a locally finite subcover $\mathcal{V}$. Take $\mathcal{W} \subseteq \mathcal{V}$ to be the $W \in \mathcal{V}$ such that $W \subseteq U_a$ for some a. Then $\mathcal{W}$ is still locally finite.

We take $U = \bigcup_{W \in \mathcal{W}} W$ and $V = X \setminus \overline{U}$. We know $b \in V$ since $\overline{U} = \bigcup \overline{W}$, and $b \notin \overline{W}$ since $W \subseteq U_a$ and b has a neighborhood (V_a) disjoint from U_a. ■

Recall that the **support** of a conitnuous function $f : X \longrightarrow \mathbb{R}$ is $\mathrm{supp}(f) = \overline{f^{-1}(\mathbb{R} \setminus \{0\})}$.

Definition 34.4. Let $\mathcal{U} = \{U_\alpha\}$ be a cover of X. A **partition of unity** subordinate to $\mathcal{U}$ is a collection $\varphi_\alpha : X \longrightarrow [0,1]$ of continuous functions such that

(1) $\operatorname{supp}(\varphi_\alpha) \subseteq U_\alpha$
(2) the collection $\operatorname{supp}(\varphi_\alpha)$ is locally finite
(3) we have $\sum_\alpha \varphi_\alpha = 1$. Note that, when evaluated at some $x \in X$, this sum is always finite by the local finite assumption (2).

Theorem 34.5. *Let X be paracompact Hausdorff, and let $\mathcal{U} = \{U_\alpha\}$ be an open cover. Then there exists a partition of unity subordinate to $\mathcal{U}$.*

Lemma 34.6 (Lee, 4.84)**.** *There exists a locally finite refinement $\{V_\alpha\}$ of $\{U_\alpha\}$ with $\overline{V_\alpha} \subseteq U_\alpha$.*

Proof of Theorem. We apply the lemma twice to get locally finite covers $\{V_\alpha\}$ and $\{W_\alpha\}$ with $\overline{W_\alpha} \subseteq V_\alpha \subseteq \overline{V_\alpha} \subseteq U_\alpha$. For each α, we use Urysohn's lemma to get $f_\alpha : X \longrightarrow [0,1]$ with $f_\alpha \equiv 1$ on $\overline{W_\alpha}$ and $\operatorname{supp}(f_\alpha) \subseteq \overline{V_\alpha} \subseteq U_\alpha$. Since $\{V_\alpha\}$ is locally finite, we can define $f : X \longrightarrow [0,1]$ by $f = \sum_\alpha f_\alpha$. Locally around some $x \in X$, the function f is a finite sum of f_α's, and so is continuous. It only remains to normalize our f_α's. Note that at any $x \in X$, we can find an α for which $x \in W_\alpha$, and so $f(x) \geq f_\alpha(x) = 1$. Thus it makes sense to define $\varphi_\alpha : X \longrightarrow [0,1]$ by

$$\varphi_\alpha(x) = \frac{f_\alpha(x)}{f(x)}.$$

We have $\operatorname{supp}(\varphi_\alpha) = \operatorname{supp}(f_\alpha)$, and so the φ_α give a partition of unity. ■

Partitions of unity come up often in the theory of vector bundles. For us, they have an powerful consequence for a very important class of spaces.