Given our discussion of continuous maps between metric spaces, it should be clear what the right notion is for maps between topological spaces.

Definition 5.1. A function $f : X \longrightarrow Y$ between topological spaces is said to be **continuous** if for every open subset $V \subseteq Y$, the preimage $f^{-1}(V)$ is open in X.

Example 5.2. Let $X = \{1, 2\}$ with topology $\mathcal{T}_X = \{\emptyset, \{1\}, X\}$ and let $Y = \{1, 2, 3\}$ with topology $\mathcal{T}_Y = \{\emptyset, \{2\}, \{3\}, \{2, 3\}, Y\}$. Which functions $X \longrightarrow Y$ are continuous?

Let's start with the open set $\{2\} \subseteq Y$. The preimage must be open, so it can either be $\emptyset$ or $\{1\}$ or X. If the preimage is X, the function is constant at 2, which is continuous.

Suppose the preimage is $\emptyset$. Then the preimage of $\{3\}$ can be either $\emptyset$ or $\{1\}$ or X. If it is $\emptyset$, we are looking at the constant function at 1, which is continuous. If $f^{-1}(3) = X$, then f is constant at 3, which is continuous. Finally, if $f^{-1}(3) = \{1\}$, then f must be the continuous function $f(1) = 3$, $f(2) = 1$.

Finally, suppose $f^{-2} = \{1\}$. Then $f^{-1}(3)$ can't be $\{1\}$ or X, so the only possible continuous f has $f^{-1}(3) = \emptyset$, so that we must have $f(1) = 2$ and $f(2) = 1$.

By the way, we asserted above that constant functions are continuous. Let's prove it!

Lemma 5.3. *Let X and Y be spaces and $y \in Y$. The constant function $c_y : X \longrightarrow Y$ at y is continuous, whatever topologies we have on X and Y.*

Proof. Let $V \subseteq Y$ be open. If $y \in V$, then $c_y^{-1}(V) = X$, which is open. On the other hand, if $y \notin V$, then $c_y^{-1}(V) = \emptyset$, so again the preimage is open. ■

Proposition 5.4. *Suppose $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are continuous. Then so is their composition $g \circ f : X \longrightarrow Z$.*

Proof. Let $V \subseteq Z$ be open. Then

$$\begin{aligned}(g \circ f)^{-1}(V) &= \{x \in X \mid (g \circ f)(x) \in V\} = \{x \in X \mid g(f(x)) \in V\} \\ &= \{x \in X \mid f(x) \in g^{-1}(V)\} = \{x \in X \mid x \in f^{-1}(g^{-1}(V))\} = f^{-1}(g^{-1}(V)).\end{aligned}$$

Now g is continuous, so $g^{-1}(V)$ is open in Y, and f is continuous, so $f^{-1}(g^{-1}(V))$ is open in X. ■

Another construction we can consider with continuous functions is the idea of restricting a continuous function to a subset. For instance, the natural logarithm is a nice continuous function $\ln : (0, \infty) \longrightarrow \mathbb{R}$, but we also get a nice continuous function by considering the logarithm only on $[1, \infty)$. To have this discussion here, we should think about how a subset of a space becomes a space in its own right.

Definition 5.5. Let X be a space and let $A \subseteq X$ be a subset. We define the subspace topology on A by saying that $V \subseteq A$ is open if and only if there exists some open $U \subseteq X$ with $U \cap A = V$.

Note that the open set $U \subseteq X$ is certainly not unique.

Example 5.6.
(1) Let $A = \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$. Then the subspace topology on $A \cong \mathbb{R}$ is the usual topology on $\mathbb{R}$. Indeed, consider the usual basis for $\mathbb{R}^2$ consisting of open disks. Intersecting these with A gives open intervals. In general, intersecting a basis for X with a subset A gives a basis for A, and here we clearly get the usual basis for the standard topology. The same would be true if we started with max-metric basis (consisting of open rectangles).
(2) Let $A = (0, 1) \subseteq X = \mathbb{R}$. We claim that $V \subseteq A$ is open in the subset topology if and only if V is open as a subset of $\mathbb{R}$. Indeed, suppose that V is open in A. Then $V = U \cap (0, 1)$

for some open U in $\mathbb{R}$. But now both U and $(0,1)$ are open in $\mathbb{R}$, so it follows that their intersection is as well. The converse is clear.

Note that this statement fails for the previous example. $(0,1) \times \{0\}$ is open in A there but not open in $\mathbb{R}^2$.

(3) Let $A = (0,1]$. Then, in the subspace topology on A, every interval $(a,1]$, with $a < 1$ is an open set. A basis for this topology on A consists in the (a,b) with $0 \le a < b < 1$ and the $(a,1]$ with $0 \le a < 1$.

(4) Let $A = (0,1) \cup \{2\}$. Then the singleton $\{2\}$ is an open subset of A! A basis consists of the (a,b) with $0 \le a < b \le 1$ and the singleton $\{2\}$.

Given a subset $A \subseteq X$, there is always the inclusion function $\iota_A : A \longrightarrow X$ defined by $\iota_A(a) = a$.

Proposition 5.7. *Given a subset $A \subseteq X$ of a topological space, the inclusion ι_A is continuous. Moreover, the subspace topology on A is the coarsest topology which makes this true.*

Proof. Suppose that $U \subseteq X$ is open. Then $\iota_A^{-1}(U) = U \cap A$ is open in A by the definition of the subspace topology.

To see that this is the coarsest such topology, suppose that $\mathcal{T}'$ is a topology which makes the inclusion $\iota_A : A \longrightarrow X$. We wish to show that $\mathcal{T}'$ is finer than the subspace topology, meaning that $\mathcal{T}_A \subseteq \mathcal{T}'$, where $\mathcal{T}_A$ is the subspace topology. So let V be open in $\mathcal{T}_A$. This means there exists $U \subseteq X$ open such that $V = U \cap A = \iota_A^{-1}(U)$. Since ι_A is continuous according to $\mathcal{T}'$, it follows that V is open in $\mathcal{T}'$. ∎

Getting back to our motivational question, suppose that $f : X \longrightarrow Y$ is continuous and let $A \subseteq X$ be a subset. We define the restriction of f to A, denoted $f_{|_A}$, by

$$f_{|_A} : A \longrightarrow Y, \qquad f_{|_A}(a) = f(a).$$

Proposition 5.8. *Let $f : X \longrightarrow Y$ be continuous and suppose that $A \subseteq X$ is a subset. Then the restriction $f_{|_A} : A \longrightarrow Y$ is continuous.*

Proof. This is just the composition $f_{|_A} = f \circ \iota_A$. ∎

6. Wed, Sept. 11

So far, we only discussed the notion of open set, but there is also the complementary notion of closed set.

Definition 6.1. Let X be a space. We say a subset $W \subseteq X$ is **closed** if the complement $X \setminus W$ is open.

Note that, despite what the name may suggest, closed does *not* mean "not open". For instance, the empty set is always both open (required for any topology) and closed (because the complement, X must be open). Similarly, there are many examples of sets that are neither open nor closed (for example, the interval $[0,1)$ in the usual topology on $\mathbb{R}$).

Proposition 6.2. *Let X be a space.*

(1) *$\emptyset$ and X are both closed in X*
(2) *If W_1, W_2 are closed, then $W_1 \cup W_2$ is also closed*
(3) *If W_i are closed for all i in some index set I, then $\bigcap_{i \in I} W_i$ is also closed.*

Proof. We prove (2). The point is that

$$X \setminus (W_1 \cup W_2) = (X \setminus W_1) \cap (X \setminus W_2).$$

This equality is known as one of the DeMorgan Laws ∎

Not only does a topology give rise to a collection of closed sets satisfying the above properties, but one can also define a topology by specifying a list of closed sets satisfying the above properties.

Similarly, we can use closed sets to determine continuity.

Proposition 6.3. *Let $f : X \longrightarrow Y$. Then f is continuous if and only if the preimage of every closed set in Y is closed in X.*

Example 6.4. The "distance from the origin function" $d : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is continuous (see HW 2). Since $\{1\} \subseteq \mathbb{R}$ is closed, it follows that the sphere $S^2 = d^{-1}(1)$ is closed in $\mathbb{R}^3$. More generally, S^{n-1} is closed in $\mathbb{R}^n$.

Example 6.5. Let X be any metric space, let $x \in X$, and let $r > 0$. Then the ball

$$B_{\leq r}(x) = \{y \in X \mid d(x, y) \leq r\}$$

is closed in X.

Remark 6.6. Note that some authors use the notation $\overline{B_r(x)}$ for the closed ball. This is a bad choice of notation, since it suggests that the closure of the open ball is the closed ball. But this is not always true! For instance, consider a set (with more than one point) equipped with the discrete metric. Then $B_1(x) = \{x\}$ is already closed, so it is its own closure. On the other hand, $B_{\leq 1}(x) = X$.

Consider the half-open interval $[a, b)$. It is neither open nor closed, in the usual topology. Nevertheless, there is a closely associated closed set, $[a, b]$. Similarly, there is a closely associated open set, (a, b). Notice the containments

$$(a, b) \subseteq [a, b) \subseteq [a, b].$$

It turns out that this picture generalizes.

Let's start with the closed set. In the example above, $[a, b]$ is the *smallest closed set containing* $[a, b)$. Why should we expect such a smallest closed set to exist in general? Recall that if we intersect arbitrarily many closed sets, we are left with a closed set.

Definition 6.7. Let $A \subseteq X$ be a subset of a topological space. We define the **closure of A in X** to be

$$\overline{A} = \bigcap_{A \subset B \text{ closed}} B.$$

Dually, we have $(a, b) \subset [a, b)$, and (a, b) is the largest open set contained in $[a, b)$.

Definition 6.8. Let $A \subseteq X$ be a subset of a topological space. We define the **interior of A in X** to be

$$\mathrm{Int}(A) = \bigcup_{A \supset U \text{ open}} U.$$

The difference of these two constructions is called the **boundary of A in X**, defined as

$$\partial A = \overline{A} \setminus \mathrm{Int}(A).$$

Example 6.9. (1) From what we have already said, it follows that $\partial[a, b) = \{a, b\}$.

(2) Let $A = \{1/n \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$. Then A is not open, since no neighborhood of any $1/n$ is contained in A. This also shows that $\mathrm{Int}(A) = \emptyset$. But neither is A closed, because no neighborhood of 0 is contained in the complement of A. This implies that $0 \in \overline{A}$, and it turns out that $\overline{A} = A \cup \{0\}$. Thus $\partial A = \overline{A} = A \cup \{0\}$.

(3) Let $\mathbb{Q} \subseteq \mathbb{R}$. Similarly to the example above, $\mathrm{Int}(\mathbb{Q}) = \emptyset$. But since $\mathbb{R} \setminus \mathbb{Q}$ does not entirely contain any open intervals, it follows that $\overline{\mathbb{Q}} = \mathbb{R}$. (A subset $A \subseteq X$ is said to be **dense** in X if $\overline{A} = X$.) Thus $\partial\mathbb{Q} = \mathbb{R} \setminus \emptyset = \mathbb{R}$.

There is a convenient characterization of the closure, which we were implicitly using above.

Proposition 6.10. *Let $A \subseteq X$. Then $x \in \overline{A}$ if and only if every neighborhood of x meets A.*

Proof. ($\Rightarrow$) Suppose $x \in \overline{A}$. Then $x \in B$ for all closed sets B containing A. Let N be a neighborhood of x. Without loss of generality, we may suppose N is open. Now $X \setminus N$ is closed but $x \notin X \setminus N$, so this set cannot contain A. This means precisely that $N \cap A \neq \emptyset$.

($\Leftarrow$) Suppose every neighborhood of x meets A. Let $A \subset B$, where B is closed in X. Now $U = X \setminus B$ is an open set not meeting A, so it cannot be a neighborhood of x. This must mean that $x \notin X \setminus B$, or in other words $x \in B$. Since B was arbitrary, it follows that x lies in every such B. ∎

In our earlier discussion of metric spaces, we considered convergence of sequences and how this characterized continuity. This is one statement from the theory of metric spaces that will not carry over to the generality of topological spaces.

Definition 6.11. We say that a sequence x_n in X converges to x in X if every neighborhood of x contains a tail of (x_n).

The following result follows immediately from the previous characterization of the closure.

Proposition 6.12. *Let (a_n) be a sequence in $A \subseteq X$ and suppose that $a_n \to x \in X$. Then $x \in \overline{A}$.*

However, the converse fails in general. To see this, consider $\mathbb{R}$ equipped with the *cocountable* topology. Recall that this means that the nonempty open subsets are the cocountable ones.

Lemma 6.13. *Suppose that $x_n \to x$ in the cocountable topology on $\mathbb{R}$. Then (x_n) is eventually constant.*

Proof. Write B for the set
$$B = \{x_n \mid x_n \neq x\}.$$
Certainly B is countable, so it is closed. By construction, $x \notin B$, so $N = X \setminus B$ is an open neighborhood of x. But $x_n \to x$, so a tail of this sequence must lie in N. Since $\{x_n\} \cap N = \{x\}$, this means that a tail of this sequence is constant, in other words, the sequence is eventually constant. ∎

7. Fri, Sept. 13

Now consider $A = \mathbb{R} \setminus \{0\} \subseteq \mathbb{R}$ in the cocountable topology. A is not closed since the only closed proper subsets are the countable ones. It follows that A must be dense in $\mathbb{R}$. However, no sequence in A can converge to 0 since a convergent sequence must be eventually constant.

Similarly, we cannot use convergence of sequences to test for continuity in general topological spaces. For instance, consider the identity map
$$\mathrm{id} : \mathbb{R}_{\text{cocountable}} \longrightarrow \mathbb{R}_{\text{standard}},$$
where the domain is given the cocountable topology and the codomain is given the usual topology. This is not continuous, since the interval $(0, 1)$ is open in $\mathbb{R}_{\text{standard}}$ but not in $\mathbb{R}_{\text{cocountable}}$. On the other hand, the identity function takes convergent sequences in $\mathbb{R}_{\text{cocountable}}$, which are necessarily eventually constant, to convergent sequences in $\mathbb{R}_{\text{standard}}$. This follows from the following result, whose proof was given back in Proposition 2.8.

Proposition 7.1. *Let $f : X \longrightarrow Y$ be continuous. If $x_n \to x$ in X then $f(x_n) \to f(x)$ in Y.*

However, all hope is not lost, since the following is true.

Proposition 7.2. *Let $f : X \longrightarrow Y$. Then f is continuous if and only if*

$$f(\overline{A}) \subseteq \overline{f(A)}$$

for every subset $A \subseteq X$.

Proof. ($\Rightarrow$) Assume f is continuous. Since $\overline{f(A)}$ is the intersection of *all* closed sets containing $f(A)$, it suffices to show that if B is such a closed set, then $f(\overline{A}) \subseteq B$. Well, $f(A) \subseteq B$, so

$$A = f^{-1}(f(A)) \subseteq f^{-1}(B).$$

Now f is continuous and B is closed, so by definition of the closure, we must have

$$\overline{A} \subseteq f^{-1}(B).$$

Applying f then gives $f(\overline{A}) \subseteq f(f^{-1}(B)) \subseteq B$.

($\Leftarrow$) Suppose that the above subset inclusion holds, and let $B \subseteq Y$ be closed. Let $A = f^{-1}(B)$. We wish to show that A is closed, i.e. that $\overline{A} = A$. Since $f(f^{-1}(B)) \subseteq B$, we know that

$$f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{B} = B.$$

Applying f^{-1} gives

$$\overline{A} = f^{-1}(f(\overline{A})) \subseteq f^{-1}(B) = A.$$

It follows that A is closed. ■

Ok, so we have learned that points in $\overline{A}$ are good enough to determine continuity of functions, but these points are not necessarily limits of sequences in A. It turns out that there is an alternative characterization of these points.

Definition 7.3. Let X be a space and $A \subseteq X$. A point $x \in X$ is said to be an **accumulation point** (or **cluster point** or **limit point**) of A if

every neighborhood of x contains a point of A other than x itself.

We sometimes write A' for the set of accumulation points of A.

Example 7.4.
(1) Let $A = (0, 1) \subseteq \mathbb{R}$. Then $A' = [0, 1]$.
(2) Let $A = \{0, 1\} \subseteq \mathbb{R}$. Then $A' = \emptyset$.
(3) Let $A = [0, 1) \cup \{2\}$. Then $A' = [0, 1]$.
(4) Let $A = \{1/n\} \subseteq \mathbb{R}$. Then $A' = \{0\}$.

The following result follows immediately from our neighborhood characterization of the closure of a set.

Proposition 7.5. *A point x is an acc. point of A if and only if $x \in \overline{A \setminus \{x\}}$.*

Certainly $A \setminus \{x\} \subseteq A$, and the closure operation preserves containment, so it follows that $A' \subseteq \overline{A}$. From the previous examples, we see that this need not be an equality. We also have $A \subseteq \overline{A}$, and it follows that

$$A \cup A' \subseteq \overline{A}.$$

Proposition 7.6. *For any subset $A \subseteq X$, we have*

$$A \cup A' = \overline{A}.$$

Proof. It remains to show that every point in the closure is either in A or in A'. Let $x \in \overline{A}$, but suppose that $x \notin A$. Then, by the neighborhood criterion, we have that for every neighborhood N of x, $N \cap A \neq \emptyset$. But since $x \notin A$, it follows that $N \cap A \setminus \{x\} \neq \emptyset$. In other words, $x \in A'$. ■

Note that, although the motivation came from looking at sequences, there is no direct relation between accumulation points of A and limits of sequences in A.

Question 7.7. *If (a_n) is a sequence in A and $a_n \to x$, is $x \in \overline{A}$?*

Answer. No. Take $A = \{x\}$ and $a_n = x$. But, if we require that $x \notin A$, then the answer is yes.

We already saw an example of a point in the closure which is not the limit of a sequence.

As the example $X = \mathbb{R}^n$ suggests, this sequences and closed sets are much better behaved for metric spaces.

Proposition 7.8. *Let $A \subseteq X$ and suppose that X is a metric space. Then $x \in \overline{A}$ if and only if x is the limit of a sequence in A.*

Proof. Suppose $a_n \to x$. Then either $x \in A'$, in which case we are done. Otherwise, there must be a neighborhood N of x such that $N \cap A \setminus \{x\} = \emptyset$. But $N \cap \{a_n\} \neq \emptyset$, so it must be that $x = a_n$ for some n, in other words $x \in A$.

On the other hand, suppose $x \in \overline{A}$. If $x \in A$, we can just take a constant sequence, so suppose not. For each n, $B_{1/n}(x)$ is a neighborhood of x, and $x \in \overline{A}$, so $B_{1/n}(x) \cap A \neq \emptyset$. Let $a_n \in B_{1/n}(x) \cap A$. Then the sequence $a_n \to x$, and $a_n \in A$ by construction. ■