11. Mon, Sept. 23

Last time, we introduced two topologies on a product $\prod_i X_i$ of spaces. The **box topology** has basis consisting of arbitrary products $\prod_i U_i$, where $U_i \subseteq X_i$ is open. But as we saw, this has too many open sets. As an example of this, we saw that the diagonal map $\mathbb{R} \longrightarrow \mathbb{R}^{\mathbb{N}}$ was not continuous. The second topology, the **product topology**, was given by the subbasis consisting of the $p_i^{-1}(U_i)$. This is coarser than the box topology if I is infinite. We left as an exercised the verification that this has the universal property we would want, namely, that if Z is any space with continuous maps $f_i : Z \longrightarrow X_i$ for each i in the indexing set, we want to have a (unique) continuous map $f : Z \longrightarrow \prod_i X_i$ having the f_i as coordinate maps.

Example 11.1. We mentioned above that the set of sequences in $\mathbb{R}$ is the infinite product $\prod_n \mathbb{R}$. What does a neighborhood of a sequence (x_n) look like in this topology? We are only allowed to constrain finitely many coordinates, so a neighborhood consists of all sequences that are near to (x_n) in some fixed, finitely many coordinates.

Example 11.2. Especially for those of you (I hope all!) that continue to 651 next semester, the spaces $I \times I$ and $S^1 \times S^1$ will be important. The first is a product of subspaces of $\mathbb{R}$. By a homework exercise for this week, this is equivalent to the subspace topology coming from $I \times I \subseteq \mathbb{R}^2$. Similarly, the product space $T^2 = S^1 \times S^1$ (a **torus**) can simply be thought of as a subspace of $\mathbb{R}^4$.

Proposition 11.3. *Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_i \overline{A_i} = \overline{\prod_i A_i}$$

in both the product and box topologies.

Proof. As usual, we have two subsets of $\prod_i X_i$ we want to show are the same, so we establish that each is a subset of the other. The following proof works in both topologies under consideration.

($\subseteq$) Let $(x_i) \in \prod \overline{A_i}$. We use the neighborhood criterion of the closure to show that $(x_i) \in \overline{\prod_i A_i}$. Thus let $U = \prod_i U_i$ be a basic open neighborhood of (x_i). Then for each j, U_j is a neighborhood of x_j. Since $x_j \in \overline{A_j}$, it follows that U_j must meet A_j in some point, say y_j. It then follows that $(y_i) \in U \cap \prod_i A_i$. By the neighborhood criterion, it follows that $(x_i) \in \overline{\prod_i A_i}$.

$\supseteq$) Now suppose that $(x_i) \in \overline{\prod_i A_i}$. For each j, let U_j be a neighborhood of x_j. Then $p_j^{-1}(U_j)$ is a neighborhood of (x_i), so it must meet $\prod_i A_i$. But this means precisely that U_j meets A_j. It follows that $x_j \in \overline{A_j}$ for all j. ■

Note that this implies that an (arbitrary) product of closed sets is closed, using either the product or box topologies. In particular, I^2 is closed in $\mathbb{R}^2$ and T^2 is closed in $\mathbb{R}^4$.

Proposition 11.4. *Suppose X_i is Hausdorff for each $i \in I$. Then so is $\prod_i X_i$ in both product and box topologies.*

Proof. Let $(x_i) \neq (x'_i) \in \prod_i X_i$. Then $x_j \neq x'_j$ for some particular j. Since X_j is Hausdorff, we can find disjoint neighborhoods U and U' of x_j and x'_j in X_j. Then $p_j^{-1}(U) = U \times \prod_{i \neq j} X_i$ and $p_j^{-1}(U') = U' \times \prod_{i \neq j} X_i$ are disjoint neighborhoods of (x_i) and (x'_i) in the product topology, so $\prod_i X_i$ is Hausdorff in the product topology.

For the box topology, we can either say that the above works just as well for the box topology, or we can say that since the box topology is a refinement of the product topology and the product topology is Hausdorff, it follows that the box topology must also be Hausdorff. ■

The converse is true as well. To see this, we use the fact that a subspace of a Hausdorff space is Hausdorff. How do we view X_j as a subspace of $\prod_i X_i$? We can think about an axis inclusion. Thus pick $y_i \in X_i$ for $i \neq j$. We define

$$a_j : X_j \longrightarrow \prod_i X_i = X_j \times \prod_{i \neq j} X_i$$

by

$$a_j(x) = (x, (y_i)).$$

Note that, by the universal property of the product, in order to check that a_j is continuous, it suffices to check that each coordinate map is continuous. But the coordinate maps are the identity and a lot of constant maps, all of which are certainly continuous. The map a_j is certainly injective (assuming all X_i are nonempty!), and it is an example of an embedding.

Definition 11.5. A map $f : X \longrightarrow Y$ is said to be an **embedding** if it is a homeomorphism onto its image $f(X)$, equipped with the subspace topology.

We already discussed injectivity and continuity of the axis inclusion a_j, so it only remains to show this is open, as a map to $a_j(X_j)$. Let $U \subseteq X_j$ be open. Then

$$a_j(U) = U \times \{(y_i)\} = p_j^{-1}(U) \cap a_j(X_j),$$

so $a_j(U)$ is open in the subspace topology on $a_j(X_j)$.

We will often do the above sort of exercise: if we introduce a new property or construction, we will ask how well this interacts with other constructions/properties.

12. Wed., Sept. 25

Towards the end of class last time, we showed that if X_i is Hausdorff for each $i \in I$, then $\prod_i X_i$ is Hausdorff. Furthermore, we said that the converse holds as well, but we were not careful enough. The argument was that we have the axis inclusion $X_i \to \prod_i X_i$, which embeds X_i as a subspace of the Hausdorff space $\prod_i X_i$. We need one additional assumption to make this work: namely, that all of the X_i are nonempty!

We introduced the concept of an embedding at the end of class and saw that the axis inclusion in a product is an example (if all X_i are nonempty!). Here is another example.

Example 12.1. Let $f : X \longrightarrow Y$ be continuous and define the graph of f to be

$$\Gamma(f) = \{(x, y) \mid y = f(x)\} \subseteq X \times Y.$$

The function

$$\gamma : X \longrightarrow X \times Y, \qquad \gamma(x) = (x, f(x))$$

is an embedding with image $\Gamma(f)$.

Let us verify that this is indeed an embedding. Injectivity is easy (this follows from the fact that one of the coordinate maps is injective), and continuity comes from the universal property for the product $X \times Y$ since id_X and f are both continuous. Note that $(p_Y)_{|\Gamma(f)}$, which is continuous since it is the restriction of the continuous projection p_Y, provides an inverse to γ.

The following construction will often be useful.

Proposition 12.2. *Let $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$ be continuous. Then the product map $f \times f' : X \times X' \longrightarrow Y \times Y'$ is also continuous.*

Proof. This follows very easily from the universal property. If we want to map continuously to $Y \times Y'$, it suffices to specify continuous maps to Y and Y'. The continuous map $X \times X' \longrightarrow Y$ is the composition

$$X \times X' \xrightarrow{p_X} X \xrightarrow{f} Y,$$

and the other needed map is the composition

$$X \times X' \xrightarrow{p_{X'}} X' \xrightarrow{f'} Y'.$$

■

What happens if we turn all of the arrows around in the defining property of a product? We might call such a thing a "coproduct". To be precise we would want a space that is universal among spaces equipped with maps *from* X and Y. In other words, given a space Z and maps $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$, we would want a unique map from the coproduct to Z, making the following diagram commute.

$$\begin{tikzcd}
X \arrow[dr] \arrow[drr, bend left, "f"] & & \\
& X \amalg Y \arrow[r, dashed, "\exists! h"] & Z \\
Y \arrow[ur] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

The glueing lemma gave us exactly such a description, in the case that our domain space X was made up of *disjoint* open subsets A and B. In general, the answer here is given by the **disjoint union**.

Recall that, as a set, the disjoint union of sets X and Y is the subset

$$X \amalg Y \subseteq (X \cup Y) \times \{1, 2\},$$

where $X \amalg Y = (X \times \{1\}) \cup (Y \times \{2\})$. More generally, given sets X_i for $i \in I$, their disjoint union $\coprod_i X_i$ is the subset

$$\coprod_i X_i \subseteq \left(\bigcup_i X_i \right) \times I$$

given by

$$\coprod_i X_i = \bigcup_i (X_i \times \{i\}).$$

There are natural inclusions $\iota_X : X \longrightarrow X \amalg Y$ or more generally $\iota_{X_i} : X_i \hookrightarrow \coprod_i X_i$. We topologize the coproduct by giving it the finest topology such that all ι_{X_i} are continuous. In other words, a subset $U \subseteq \coprod_i X_i$ is open if and only if $U \cap X_i$ is open for all i.

Note that in the case of a coproduct of two spaces, the subspace topology on $X \subseteq X \amalg Y$ agrees with the original topology on X. Furthermore, both X and Y are open in $X \amalg Y$, so the universal property for the coproduct is precisely the glueing lemma.

Example 12.3. (1) Consider $X = [0,1]$ and $Y = [2,3]$. Then in this case $X \amalg Y$ is homeomorphic to the subspace $X \cup Y$ of $\mathbb{R}$. The same is true of these two intervals are changed to be open or half-open.

(2) Consider $X = (0,1)$ and $Y = \{1\}$. Then $X \amalg Y$ is **not** homeomorphic to $(0,1) \cup \{1\} = (0,1]$. The singleton $\{1\}$ is open in $X \amalg Y$ but not in $(0,1]$. Instead, $X \amalg Y$ is homeomorphic to $(0,1) \cup \{2\}$.

(3) Similarly $(0,1) \amalg [1,2]$ is homeomorphic to $(0,1) \cup [2,3]$ but not to $(0,1) \cup [1,2] = (0,2]$.

(4) In yet another similar example, $(0,2) \amalg (1,3)$ is homeomorphic to $(0,1) \cup (2,3)$ but not to $(0,2) \cup (1,3) = (0,3)$.

13. Fri, Sept. 27

Proposition 13.1. *Let X_i be spaces, for $i \in I$. Then $\coprod_i X_i$ is Hausdorff if and only if all X_i are Hausdorff.*

Proof. This is even easier than for products. First, X_i always embeds as a subspace of the coproduct, so it follows that X_i is Hausdorff if the coproduct is as well. On the other hand, suppose all X_i are Hausdorff and suppose that $x \neq y$ are points of $\coprod_i X_i$. Either x and y come from different X_i's, in which case the X_i's themselves serve as the disjoint neighborhoods. The alternative is that x and y live in the same Hausdorff X_i, but then we can find disjoint neighborhoods in X_i. ■

The next important construction is that of a quotient, or identification space.

The general setup is that we have a surjective map $q : X \longrightarrow Y$, which we view as making an identification of points in X. More precisely, suppose that we have an equivalence relation $\sim$ on X. We can consider the set $X/\sim$ of equivalence classes in X. There is a natural surjective map $q : X \longrightarrow X/\sim$ which takes $x \in X$ to its equivalence class.

And in fact every surjective map is of this form. Suppose that $q : X \longrightarrow Y$ is surjective. We define a relation on X by saying that $x \sim x'$ if and only if $q(x) = q(x')$. Then the function $X/\sim \longrightarrow Y$ sending the class of x to $q(x)$ is a bijection.

We want to mimic the above situation in topology, but to understand what this should mean, we first look at the universal property of the quotient for sets. This says: if $f : X \longrightarrow Z$ is a function that is constant on the equivalence classes in X, then there is a (unique) factorization $g : X/\sim \longrightarrow Z$ with $g \circ q = f$.

We want to have a similar setup in topology. Said in the equivalence relation framework, given a space X and a relation $\sim$ on X, we want a continuous map $q : X \longrightarrow Y$ such that given any space Z with a continuous map $f : X \longrightarrow Z$ which is constant on equivalence classes, there is a unique continuous map $g : Y \longrightarrow Z$ such that $g \circ q = f$. By considering the cases in which Z is a set with the trivial topology, so that maps to Z are automatically continuous, we can see that on the level of sets $q : X \longrightarrow Y$ must be $X \longrightarrow X/\sim$. It remains only to specify the topology on $Y = X/\sim$.

We want the topological quotient to be the universal example of a continuous map out of X which is constant on equivalence classes. Universal here means that we always want to have a map $Y \longrightarrow Z$ whenever $f : X \longrightarrow Z$ is another such map. Since we want to construct maps *out of* Y, this suggests we should include as many open sets as possible in Y. This leads to the following definition.

Definition 13.2. We say that a surjective map $q : X \longrightarrow Y$ is a **quotient map** if $V \subseteq Y$ is open if and only if $q^{-1}(V)$ is open in X.

One implication is the definition of continuity, but the other is given by our desire to include as many opens as we can.

Proposition 13.3. *(Universal property of the quotient) Let $q : X \longrightarrow Y$ be a quotient map. If Z is any space, and $f : X \longrightarrow Z$ is any continuous map that is constant on the fibers*[1] *of q, then there exists a unigue continuous $g : Y \longrightarrow Z$ such that $g \circ q = f$.*

Proof. It is clear how g must be defined: $g(y) = f(x)$ for any $x \in q^{-1}(y)$. It remains to show that g is continuous. Let $W \subseteq Z$ be open. We want $g^{-1}(W) \subseteq Y$ to be open as well. By the definition of a quotient map, $g^{-1}(W)$ is open if and only if $q^{-1}(g^{-1}(W)) = (g \circ q)^{-1}(W) = f^{-1}(W)$ is open, so we are done by continuity of f. ■

Example 13.4. Define $q : \mathbb{R} \longrightarrow \{-1, 0, 1\}$ by

$$q(x) = \begin{cases} 0 & x = 0 \\ \frac{|x|}{x} & x \neq 0. \end{cases}$$

What is the resulting topology on $\{-1, 0, 1\}$? The points -1 and 1 are open, and the only open set containing 0 is the whole space.

Note that this example shows that a quotient of a Hausdorff space need not be Hausdorff.

Proposition 13.5. *Let $q : X \longrightarrow Y$ be a continuous, surjective, open map. Then q is a quotient map. The same is true if q is closed instead of open.*

Proof. One implication is simply the definition of continuity. For the other, suppose that $V \subseteq Y$ is a subset such that $q^{-1}(V) \subseteq X$ is open. Then $q(q^{-1}(V))$ is open since q is open. Finally, we have $V = q(q^{-1}(V))$ since q is surjective. ■

The converse is not true, however, as the next example shows.

Example 13.6. Consider $q; \mathbb{R} \longrightarrow [0, \infty)$ given by

$$q(x) = \begin{cases} 0 & x \leq 0 \\ x & x \geq 0. \end{cases}$$

The quotient topology on $[0, \infty)$ is the same as the subspace topology it gets from $\mathbb{R}$. But this is not an open map, since the image of $(-2, -1)$ is $\{0\}$, which is not open.

[1] A "fiber" is simply the preimage of a point.