20. Mon, Oct. 14

Last time, we introduced the idea of connectedness and showed (1) that the connected subsets of $\mathbb{R}$ are precisely the intervals and (2) the image of a connected space under a continuous map is connected. This implies.

Theorem 20.1 (Intermediate Value Theorem). *Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous. Then f attains ever intermediate value between $f(a)$ and $f(b)$.*

Proof. This follows from the fact that the image is an interval. ∎

We also showed that an overlapping union of connected subspaces is connected.

As an application, we get that products interact well with connectedness.

Proposition 20.2. *Assume $X_i \neq \emptyset$ for all $i \in \{1, \dots, n\}$. Then $\prod_{i=1}^{n} X_i$ is connected if and only if each X_i is connected.*

Proof. ($\Rightarrow$) This follows from Prop 19.5, as $p_i : \prod_i X_i \longrightarrow X_i$ is surjective (this uses that all X_j are nonempty).

($\Leftarrow$) Suppose each X_i is connected. By induction, it suffices to show that $X_1 \times X_2$ is connected. Pick any $z \in X_2$. We then have the embedding $X_1 \hookrightarrow X_1 \times X_2$ given by $x \mapsto (x, z)$. Since X_1 is connected, so is its image C in the product. Now for each $x_1 \in X_1$, we have an embedding $\iota_{x_1} : X_2 \hookrightarrow X_1 \times X_2$ given by $y \mapsto (x_1, y)$. Let $D_{x_1} = \iota_{x_1}(X_2) \cup C$. Note that each D is connected, being the overlapping union of two connected subsets. But we can write $X_1 \times X_2$ as the overlapping union of all of the D_{x_1}, so by the previous result the product is connected. ∎

The following result is easy but useful.

Proposition 20.3. *Let $A \subseteq B \subseteq \overline{A}$ and suppose that A is connected. Then so is B.*

Proof. Exercise ∎

Theorem 20.4. *Assume $X_i \neq \emptyset$ for all $i \in I$, where is I is arbitrary. Then $\prod_i X_i$ is connected if and only if each X_i is connected.*

Proof. As in the finite product case, it is immediate that if the product is connected, then so is each factor.

We sketch the other implication. We have already showed that each finite product is connected. Now let $(z_i) \in \prod_i X_i$. For each $j \in I$, write $D_j = p_j^{-1}(z_j) \subseteq \prod_i X_i$.

For each finite collection $j_1, \dots, j_k \in I$, let

$$F_{j_1,\dots,j_k} = \bigcap_{j \neq j_1,\dots,j_k} D_j \subseteq \prod_i X_i.$$

Then $F_{j_1,\dots,j_k} \cong X_{j_1} \times \cdots \times X_{j_k}$, so it follows that $F_{j_1,\dots,j_k}$ is connected. Now $(z_i) \in F_{j_1,\dots,j_k}$ for every such tuple, so it follows that

$$F = \bigcup F_{j_1,\dots,j_k}$$

is connected.

It remains to show that F is dense in $\prod_i X_i$ (in other words, the closure of F is the whole product). Let

$$U = p_{j_1}^{-1}(U_{j_1}) \cap \cdots \cap p_{j_k}^{-1}(U_{j_k})$$

be a nonempty basis element. Then U meets $f_{j_1,\dots,j_k}$, so U meets F. Since U was arbitrary, it follows that F must be dense. ■

Note that the above proof would not have worked with the box topology. We can show directly that $\mathbb{R}^{\mathbb{N}}$, equipped with the box topology, is not connected. Consider the subset $\mathcal{B} \subset \mathbb{R}^{\mathbb{N}}$ consisting of bounded sequences. If $(z_i) \in \mathcal{B}$, then $\prod_i (z_i - 1, z_i + 1)$ is a neighborhood of (z_i) in $\mathcal{B}$. On the other hand, if $(z_i) \notin \mathcal{B}$, the same formula gives a neighborhood consisting entirely of unbounded sequences. We conclude that $\mathcal{B}$ is a nontrivial clopen set in the box topology.

Ok, so we have looked at examples and studied this notion of being connected, but if you asked your calculus students to describe what it should mean for a subset of $\mathbb{R}$ to be connected, they probably wouldn't come up with the "no nontrivial clopen subsets" idea. Instead, they would probably say something about being able to connect-the-dots. In other words, you should be able to draw a line from one point to another while staying in the subset. This leads to the following idea.

21. Wed, Oct. 16

Definition 21.1. We say that $A \subseteq X$ is **path-connected** if for every pair a, b of points in A, there is a continuous function (a path) $\gamma : I \longrightarrow X$ with $\gamma(0) = a$ and $\gamma(1) = b$.

This is not unrelated to the earlier notion.

Proposition 21.2. *If $A \subseteq X$ is path-connected, then it is also connected.*

Proof. Pick a point $a_0 \in A$. For any other $b \in A$, we have a path γ_b in A from a_0 to b. Then the image $\gamma_b(I)$ is a connected subset of A containing both a_0 and b. It follows that

$$A = \bigcup_{b \in A} \gamma_b(I)$$

is connected, as it is the overlapping union of connected sets. ■

For subsets $A \subseteq \mathbb{R}$, we have

$$A \text{ is path-connected} \Rightarrow A \text{ is connected} \Leftrightarrow A \text{ is an interval} \Rightarrow A \text{ is path-connected.}$$

So the two notions coincide for subsets of $\mathbb{R}$. But the same is not true in $\mathbb{R}^2$!

Example 21.3 (Topologist's sine curve)**.** Let Γ be the graph of $\sin(1/x)$ for $x \in (0, \pi]$.Then Γ is homeomorphic to $(0, \pi]$ and is therefore path-connected and connected. Let C be the closure of Γ in $\mathbb{R}^2$. Then C is connected, as it is the closure of a connected subset. However, it is not path-connected (HW VI), as there is no path in C connecting the origin to the right end-point $(\pi, 0)$.

Path-connectedness has much the same behavior as connectedness.

Proposition 21.4.

(1) *Images of path-connected spaces are path-connected*
(2) *Overlapping unions of path-connected spaces are path-connected*
(3) *Finite products of path-connected spaces are path-connected*

However, the topologist's sine curve shows that closures of path-connected subsets need not be path-connected.

Our proof of connectivity of $\prod_i X_i$ last time used this closure property for connected sets, so the earlier argument does not adapt easily to path-connectedness. But it turns out to be easier to prove.

Theorem 21.5. *Assume $X_i \neq \emptyset$ for all $i \in I$, where is I is arbitrary. Then $\prod_i X_i$ is path-connected if and only if each X_i is path-connected.*

Proof. The interesting direction is ($\Leftarrow$). Thus assume that each X_i is path-connected. Let (x_i) and (y_i) be points in the product $\prod_i X_i$. Then for each $i \in \mathcal{I}$ there is a path γ_i in X_i with $\gamma_i(0) = x_i$ and $\gamma_i(1) = y_i$. By the universal property of the product, we get a continuous path

$$\gamma = (\gamma_i) : [0,1] \longrightarrow \prod_i X_i$$

with $\gamma(0) = (x_i)$ and $\gamma(1) = (y_i)$. ∎

The overlapping union property for (path-)connectedness allows us to make the following definition.

Definition 21.6. Let $x \in X$. We define the **connected component** (or simply component) of x in X to be

$$C_x = \bigcup_{\substack{x \in C \\ \text{connected}}} C.$$

Similarly, the **path-component** of X is defined to be

$$PC_x = \bigcup_{\substack{x \in P \\ \text{connected}}} P.$$

The overlapping union property guarantees that C_x is connected and that PC_x is path-connected. Since path-connected sets are connected, it follows that for any x, we have $PC_x \subseteq C_x$. An immediate consequence of the above definition(s) is that any (path-)connected subset of X is contained in some (path-)component.

Example 21.7. Consider $\mathbb{Q}$, equipped with the subspace topology from $\mathbb{R}$. Then the only connected subsets are the singletons, so $C_x = \{x\}$. Such a space is said to be **totally disconnected**.

Note that for any space X, each component C_x is closed as $\overline{C_x}$ is a connected subset containing x, which implies $\overline{C_x} \subseteq C_x$. However, the above example shows that components need not be open. The situation is worse for path-components: they need not be open or closed.

Definition 21.8. Let X be a space. We say that X is **locally connected** if any neighborhood U of any point x contains a connected neighborhood $x \in V \subset U$. SImilarly X is **locally path-connected** if any neighborhood U of any point x contains a path-connected neighborhood $x \in V \subset U$.

This may seem like a strange definition, but it has the following nice consequence.

Proposition 21.9. *Let X be a space. The following are equivalent.*

(1) *X is locally connected*
(2) *X has a basis consisting of connected open sets*

(3) *for every open set* $U \subseteq X$*, the components of* U *are open in* X

Proof. We show (1) $\Leftrightarrow$ (3).

Suppose that X is locally connected and let $U \subseteq X$ be open. Take $C \subseteq U$ to be a component. Let $x \in C$. We can then find a connected neighborhood $x \in V \subseteq U$. Since C is the component of x, we must have $V \subseteq C$, which shows that C is open.

Suppose, on the other hand, that (3) holds. Let U be a neighborhood of x. Then the component C_x of x in U is the desired neighborhood V. ■

In particular, this says that the components are open if X is locally connected.

22. Fri, Oct. 18

The locally path-connected property is even better.

Proposition 22.1. *Let* X *be a space. The following are equivalent.*

(1) X *is locally path-connected*
(2) X *has a basis consisting of path-connected open sets*
(3) *for every open set* $U \subseteq X$*, the path-components of* U *are open in* X
(4) *for every open set* $U \subseteq X$*, every component of* U *is path-connected and open in* X*.*

Proof. The implications (1) $\Leftrightarrow$ (3) are similar to the above. We argue for (1) $\Leftrightarrow$ (4).

Assume X is locally path-connected, and let C be a component of an open subset $U \subseteq X$. Let $P \subseteq C$ be a nonempty path-component. Then P is open in X. But all of the other path-components of C are also open, so their union, which is the complement of P, must be open. It follows that P is closed. Since C is connected, we must have $P = C$.

On the other hand, suppose that (4) holds. Let U be a neighborhood of x. Then the component C_x of x in U is the desired neighborhood V. ■

In particular, this says that the components and path-components agree if X is locally path-connected.

Just as path-connected implies connected, locally path-connected implies locally-connected. But, unfortunately, there are no other implications between the four properties.

Example 22.2. The topologist's sine curve is connected, but not path-connected or locally connected or locally path-connected (see HWVI). Thus it is possible to be connected but not locally so.

Example 22.3. For any space X, the **cone** on X is defined to be $CX = X \times [0,1]/X \times \{1\}$. The cone on any space is always path-connected. In particular, the cone on the topologist's sine curve is connected and-path connected but not locally connected or locally path-connected.

Example 22.4. A disjoint union of two topologist's sine curves gives an example that is not connected in any of the four ways.

Example 22.5. Note that if X is locally path-connected, then connectedness is equivalent to path-connectedness. A connected example would be $\mathbb{R}$ or a one-point space. A disconnected example would be $(0,1) \cup (2,3)$ or a two point (discrete) space.

Finally, we have spaces that are locally connected but not locally path-connected.

Example 22.6. The cocountable topology on $\mathbb{R}$ is connected and locally connected but not path-connected or locally path-connected. (See HWVI)

Example 22.7. The cone on the cocountable topology will give a connected, path-connected, locally connected space that is not locally path-connected.

Example 22.8. Two copies of $\mathbb{R}_{\text{cocountable}}$ give a space that is locally connected but not connected in the other three ways.

The next topic is one of the major ones in the course: compactness. As we will see, this is the analogue of a "closed and bounded subset" in a general space. The definition relies on the idea of coverings.

Definition 22.9. An **open cover** of X is a collection $\mathcal{U}$ of open subsets that cover X. In other words, $\bigcup_{U \in \mathcal{U}} U = X$. Given two covers $\mathcal{U}$ and $\mathcal{V}$ of X, we say that $\mathcal{V}$ is a **subcover** if $\mathcal{V} \subseteq \mathcal{U}$.

Definition 22.10. A space X is said to be **compact** if every open cover has a *finite* subcover (i.e. a cover involving finitely many open sets).

Example 22.11. Clearly any finite topological space is compact, no matter the topology.

Example 22.12. An infinite set with the discrete topology is *not* compact, as the collection of singletons gives an open cover with no finite subcover.

Example 22.13. $\mathbb{R}$ is not compact, as the open cover $\mathcal{U} = \{(-n, n) \mid n \in \mathbb{N}\}$ has no finite subcover.

Theorem 22.14. *Let $a < b$. Then $[a, b]$ is a compact subset of $\mathbb{R}$.*

Proof. Let $\mathcal{U}$ be an open cover. Then some element of the cover must contain a. Pick such an element and call it U_1.

Consider the set
$$\mathcal{E} = \{c \in [a, b] \mid [a, c] \text{ is finitely covered by } \mathcal{U}\}.$$
Certainly $a \in \mathcal{E}$ and $\mathcal{E}$ is bounded above by b. By the Least Upper Bound Axiom, $s = \sup \mathcal{E}$ exists. Note that $a \leq s \leq b$, so we must have $s \in U_s$ for some $U_s \in \mathcal{U}$. But then for any $c < s$ with $c \in U_s$, we have $c \in \mathcal{E}$. This means that
$$[a, c] \subseteq U_1 \cup \cdots \cup U_k$$
for $U_1, \ldots, U_k \in \mathcal{U}$. But then $[a, s] \subseteq U_1 \cup \cdots \cup U_k \cup U_s$. This shows that $s \in \mathcal{E}$. On the other hand, the same argument shows that for any $s < d < b$ with $d \in U_s$, we would similarly have $d \in \mathcal{E}$. Since $s = \sup \mathcal{E}$, there cannot exist such a d. This implies that $s = b$. ∎