1. (Metric completion) Let $X$ be a metric space. Let $C_X$ be the set of Cauchy sequences in $X$. Define a relation on $C_X$ by

$$(x_n) \sim (y_n) \text{ if } \lim_{n} d_X(x_n, y_n) = 0.$$ 

Define $X^* = (C_X / \sim)$. Then $d\left((x_n), (y_n)\right) = \lim_{n} d_X(x_n, y_n)$ defines a metric on $X^*$, and the map

$$\iota : X \longrightarrow X^*$$

$$x \mapsto (x, x, \ldots)$$

is an isometric embedding.

(a) Show that $\iota(X)$ is dense in $X^*$.

(b) Show that if $A$ is a dense subset of a metric space $Z$ and every Cauchy sequence in $A$ converges in $Z$, then $Z$ is complete. Conclude that $X^*$ is complete.

(c) Show that $X$ is totally bounded if and only if $X^*$ is compact.

2. (a) Find an example of a bijective continuous map $f : X \longrightarrow Y$, where $X$ is locally compact but $Y$ is not.

(b) Show that if $f : X \longrightarrow Y$ is a continuous, open surjection and $X$ is locally compact, then $Y$ must be locally compact.

3. Let $X = \mathbb{R} \times \mathbb{Z} / \sim$, where $\sim$ is the equivalence relation generated by $(x, n) \sim (x, k)$ for all $n, k \in \mathbb{Z}$ and $x \neq 0$. Show that $X$ is locally compact but does not have a basis of precompact open sets.

4. (Stereographic Projection) Let $N = (0, \ldots, 0, 1) \in S^n$ be the North Pole. Define a homeomorphism $S^n \setminus \{N\} \cong \mathbb{R}^n$ as follows. For each $x \neq N \in S^n$, consider the ray starting at $N$ and passing through $x$. This meets the equatorial hyperplane (defined by $x_{n+1} = 0$) in a point, which we call $p(x)$.

(a) Determine a formula for $p$ and show that it gives a homeomorphism.

(b) Conclude that the one-point compactification of $\mathbb{R}^n$ is $S^n$. 