26. Mon, Oct. 27

Closely related to compactness is the following notion.

Definition 26.1. We say that a space X is **sequentially compact** if every sequence in X has a convergent subsquence.

Example 26.2. The open interval $(0,1)$ is not sequentially compact because $\{1/n\}$ has no subsequence that converges in $(0,1)$. If we consider instead $[0,1]$, this example no longer works, and we will see that $[0,1]$ is indeed sequentially compact.

In general, there is no direct relation between compactness and sequential compactness.

Example 26.3. Consider $X = \prod_{[0,1]} \{0,1\}$ under the product topology. By the Tychonoff theorem, X is compact. However, it is not sequentially compact. Let $f_n \in X$ be defined by $f_n(x) =$ the nth digit in the binary expansion of x. We claim that (f_n) has no convergent subsequence. Recall that convergence in X means *pointwise* convergence of functions. Let (f_{n_k}) be any subsequence. In order for this to converge, it the sequence $f_{n_k}(x)$ would need to converge for every x. This is simply a sequence of 0's and 1's, so it must be eventually constant. But no matter the subsequence f_{n_k}, we can find an $x \in I$ whose corresponding sequence of digits is not eventually constant.

Example 26.4. Let

$$X = \left\{ x \in \prod_{\mathbb{R}} \{0,1\} \,\middle|\, x^{-1}(1) \text{ is countable.} \right\}$$

We here consider $\{0,1\}$ with the discrete topology, and X is a subspace of the product. For each $r \in \mathbb{R}$, let $B_r = \{x \in X \mid x(r) = 0\}$. This is a prebasis element and so is open. Then the collection $\{B_r\}_{r\in\mathbb{R}}$ gives an open cover of X, but it clearly has no finite subcover.

Now let (x_n) be a sequence in X. Let

$$S = \bigcup_n x_n^{-1}(1).$$

S is a countable union of countable sets, so it is countable. Let $Y = \prod_S \{0,1\}$, and let $q : X \longrightarrow Y$ be the restriction along $S \hookrightarrow \mathbb{R}$. Then $q(x_n)$ is a sequence in $Y = \{0,1\}^S$. It can be seen directly that Y is sequentially compact, so that some subsequence $q(x_{n_k})$ of $q(x_n)$ must converge to, say $y \in Y$. Let $z \in X$ be the function with $x^{-1}(1) = y^{-1}(1)$. But then x_{n_k} converges to z since each x_n is identically 0 on $\mathbb{R} \setminus S$.

We have shown that X is sequentially compact space but not compact.

There is one more form of compactness.

Definition 26.5. A space is said to be **limit point compact** of every infinite subset has a limit point (accumulation point).

Theorem 26.6. *If X is a metric space, then X is compact if and only if it is sequentually compact if and only if it is limit point compact.*

Proof. See Munkres, Theorem 28.2 or Lee, Lemmas 4.42-4.44. ∎

In $\mathbb{R}^n$, this result is knwon by the following name.

Theorem 26.7 (Bolzano-Weierstrass). *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

We also saw that the compact subsets of the metric space $\mathbb{R}^n$ are the closed and bounded ones. Do we have an analogue of this statement for an arbitrary metric space X? First, note that closed and bounded is not enough in general to guarantee compactness, as any infinite discrete metric space shows.

27. Wed, Oct. 29

We discussed the takehome exam today.

28. Fri, Oct. 31

Definition 28.1. We say that a metric space X is **totally bounded** if, for every $\epsilon > 0$, there is a finite covering of X by ϵ-balls.

It is clear that compact implies totally bounded because, for any fixed $\epsilon > 0$, the B_ϵ give an open covering. This suffices to handle the discrete metric case, as a discrete metric space is totally bounded $\iff$ it is finite $\iff$ it is compact. However, closed and totally bounded is still not enough, as $[0,1] \cap \mathbb{Q}$ is closed and totally bounded (either in $\mathbb{Q}$ or in itself) but not compact, as we have already seen. Recall that a metric space is **complete** if every Cauchy sequence converges in X.

Theorem 28.2. *Let X be metric. Then X is compact $\iff$ X is complete and totally bounded.*

Proof. ($\Rightarrow$) We have already mentioned why compactness implies totally bounded. Let (x_n) be a Cauchy sequence in X. Then, since X is sequentially compact, a subsequence of (x_n) converges. But if $x_{n_k} \to x$, then we must also have $x_n \to x$ since x_n is Cauchy (prove this)! It follows that X is complete.

($\Leftarrow$) Suppose now that X is complete and totally bounded. We show that X is sequentially compact. Let (x_n) be any sequence in X. Since X is complete, it suffices to show that (x_n) has a subsequence that is Cauchy.

For each n, we have a finite covering of X by k_n balls of radius $1/n$. Start with $n = 1$. One of these balls must contain infinitely many x_n's and so a subsequence of (x_n). Now cover X by finitely many balls of radius $1/2$. Again, one of these contains a subsequence of the previous subsequence. We continue in this way ad infinitum. We obtain the desired Cauchy subsequence as follows. First, pick x_{n_1} to be in our original subsequence (in the chosen ball of radius 1). Then pick x_{n_2} to be in the subsubsequence in our chosen ball of radius $1/2$ (and pick it such that $n_2 > n_1$. After (many, many) choices, we get a subsequence of x_n such that $\{x_{n_k}\}_{k\geq m}$ is contained in a ball of radius $1/m$. It follows that x_{n_k} is Cauchy. ∎

Note that $[0,1] \cap \mathbb{Q}$ is not complete, as the sequence

$$x_n = \text{ the decimal expansion of } 1/\pi \text{ cut off after the } n\text{th digit}$$

is a Cauchy sequence in $[0,1] \cap \mathbb{Q}$ which does not converge.

Definition 28.3. We say that a space is **locally compact** if every $x \in X$ has a compact neighborhood (recall that we do not require neighborhoods to be open).

This looks different from our other "local" notions. To get a statement in the form we expect, we introduce more terminology $A \subseteq X$ is **precompact** if $\overline{A}$ is compact.

Proposition 28.4. *Let X be Hausdorff. TFAE*

(1) *X is locally compact*
(2) *every $x \in X$ has a precompact neighborhood*
(3) *X has a basis of precompact open sets*

Proof. It is clear that (3) $\Rightarrow$ (2) $\Rightarrow$ (1) without the the Hausdorff assumption, so we show that (1) $\Rightarrow$ (3). Suppose X is locally compact and Hausdorff. Let V be open in X and let $x \in V$. We want a precompact open neighborhood of x in V. Since X is locally compact, we have a compact neighborhood K of x, and since X is Hausdorff, K must be closed. Since V and K are both neighborhoods of x, so is $V \cap K$. Thus let $x \in U \subseteq V \cap K$. Then $\overline{U} \subseteq K$ since K is closed, and $\overline{U}$ is compact since it is a closed subset of a compact set. ■

In contrast to the local connectivity properties, it is clear that any compact space is locally compact. But this is certainly a generalization of compactness, since *any interval* in $\mathbb{R}$ is locally compact.

Example 28.5. A standard example of a space that is not locally compact is $\mathbb{Q} \subseteq \mathbb{R}$. We show that 0 does not have any compact neighborhoods . Let V be any neighborhood of 0. Then it must contain $(-\pi/n, \pi/n)$ for some n. Now

$$\mathcal{U} = \left\{ \left(-\pi/n, \left(\frac{k}{k+1} \right) \pi/n \right) \right\} \cup \left\{ V \cap (\pi/n, \infty), V \cap (-\infty, -\pi/n) \right\}$$

is an open cover of V with no finite subcover.

Remark 28.6. Why did we define local compactness in a different way from local (path)-connectedness? We could have defined locally connected to mean that every point has a connected neighborhood , which follows from the actual definition. But then we would not have that locally connected is equivalent to having a basis of connected open sets. On the other hand, we could try the $x \in K \subseteq U$ version of locally compact, but of course we don't want to allow $K = \{x\}$, so the next thing to require is $x \in V \subseteq U$, where V is precompact. As we showed in Prop 28.4, this is equivalent to our definition of locally compact in the presence of the Hausdorff condition. Without the Hausdorff condition, compactness does not behave quite how we expect.