29. Mon, Nov. 3

Locally compact Hausdorff spaces are a very nice class of spaces (almost as good as compact Hausdorff). In fact, any such space is close to a compact Hausdorff space.

**Definition 29.1.** A compactification of a noncompact space $X$ is an embedding $i : X \hookrightarrow Y$, where $Y$ is compact and $i(X)$ is dense.

We will typically work with Hausdorff spaces $X$, in which case we ask the compactification $Y$ to also be Hausdorff.

**Example 29.2.** The open interval $(0,1)$ is not compact, but $(0,1) \hookrightarrow \mathbb{R}$ is a compactification. Note that the exponential map $\exp : (0,1) \rightarrow S^1$ also gives a (different) compactification.

There is often a smallest compactification, given by the following construction.

**Definition 29.3.** Let $X$ be a space and define $\hat{X} = X \cup \{\infty\}$, where $U \subseteq \hat{X}$ is open if either

- $U \subseteq X$ and $U$ is open in $X$ or
- $\infty \in U$ and $\hat{X} \setminus U \subseteq X$ is compact.

**Proposition 29.4.** Suppose that $X$ is Hausdorff and noncompact. Then $\hat{X}$ is a compactification. If $X$ is locally compact, then $\hat{X}$ is Hausdorff.

**Proof.** We first show that $\hat{X}$ is a space! It is clear that both $\emptyset$ and $\hat{X}$ are open.

Suppose that $U_1$ and $U_2$ are open. We wish to show that $U_1 \cap U_2$ is open.

- If neither open set contains $\infty$, this follows since $X$ is a space.
- If $\infty \in U_1$ but $\infty \notin U_2$, then $K_1 = X \setminus U_1$ is compact. Since $X$ is Hausdorff, $K_1$ is closed in $X$. Thus $X \setminus K_1 = U_1 \setminus \{\infty\}$ is open in $X$, and it follows that $U_1 \cap U_2 = (U_1 \setminus \{\infty\}) \cap U_2$ is open.
- If $\infty \in U_1 \cap U_2$, then $K_1 = X \setminus U_1$ and $K_2 = X \setminus U_2$ are compact. It follows that $K_1 \cup K_2$ is compact, so that $U_1 \cap U_2 = X \setminus (K_1 \cup K_2)$ is open.
- Suppose we have a collection $U_i$ of open sets. If none contain $\infty$, then neither does $\bigcup_i U_i$, and the union is open in $X$. If $\infty \in U_j$ for some $j$, then $\infty \in \bigcup_i U_i$ and

$$\hat{X} \setminus \bigcup_i U_i = \bigcap_i (\hat{X} \setminus U_i) = \bigcap_i (X \setminus U_i)$$

is a closed subset of the compact set $X \setminus U_j$, so it must be compact.

Next, we show that $i : X \rightarrow \hat{X}$ is an embedding. Continuity of $i$ again uses that compact subsets of $X$ are closed. That $i$ is open follows immediately from the definition of $\hat{X}$.

To see that $i(X)$ is dense in $\hat{X}$, it suffices to see that $\{\infty\}$ is not open. But this follows from the definition of $\hat{X}$, since $X$ is not compact.

Finally, we show that $\hat{X}$ is compact. Let $\mathcal{U}$ be an open cover. Then some $U \in \mathcal{U}$ must contain $\infty$. The remaining elements of $\mathcal{U}$ must cover $X \setminus U$, which is compact. It follows that we can cover $X \setminus U$ using only finitely many elements, so $\mathcal{U}$ has a finite subcover.

Now suppose that $X$ is locally compact. Let $x_1$ and $x_2$ in $\hat{X}$. If neither is $\infty$, then we have disjoint neighborhoods in $X$, and these are still disjoint neighborhoods in $\hat{X}$. If $x_2 = \infty$, let $x_1 \in U \subseteq K$, where $U$ is open and $K$ is compact. Then $U$ and $V = \hat{X} \setminus K$ are the desired disjoint neighborhoods.

**Example 29.5.** We saw that $S^1$ is a one-point compactification of $(0,1) \cong \mathbb{R}$. You will show on your homework that similarly $S^n$ is a one-point compactification of $\mathbb{R}^n$.
Example 29.6. As we have seen, $\mathbb{Q}$ is not locally compact, so we do not expect $\hat{\mathbb{Q}}$ to be Hausdorff. Indeed, the point $\infty$ is dense in $\hat{\mathbb{Q}}$. Because of the topology on $\hat{\mathbb{Q}}$, this is equivalent to showing that for any open, nonempty subset $U \subseteq \mathbb{Q}$, $U$ is not contained in any compact subset. Since $\mathbb{Q}$ is Hausdorff, if $U$ were contained in a compact subset, then $\overline{U}$ would also be compact. But as we have seen, for any interval $(a, b) \cap \mathbb{Q}$, the closure in $\mathbb{Q}$, which is $[a, b] \cap \mathbb{Q}$, is not compact.

30. Wed, Nov. 5

Next, we show that the situation we observed for compactifications of $(0, 1)$ holds quite generally.

**Proposition 30.1.** Let $X$ be locally compact Hausdorff and let $f : X \to Y$ be a compactification. Then there is a (unique) quotient map $q : Y \to \hat{X}$ such that $q \circ f = \iota$.

We will need:

**Lemma 30.2.** Let $X$ be locally compact Hausdorff and $f : X \to Y$ a compactification. Then $f$ is open.

*Proof.* Since $f$ is an emebedding, we can pretend that $X \subseteq Y$ and that $f$ is simply the inclusion. We wish to show that $X$ is open in $Y$. Thus let $x \in X$. Let $U$ be a precompact neighborhood of $x$. Thus $K = \text{cl}_X(U)$ is compact and so must be closed in $Y$ (and $X$) since $Y$ is Hausdorff. By the definition of the subspace topology, we must have $U = V \cap X$ for some open $V \subseteq Y$. Then $V$ is a neighborhood of $x$ in $Y$, and $V = V \cap Y = V \cap \text{cl}_Y(X) \subseteq \text{cl}_Y(V \cap X) = K \subseteq X$. ■

*Proof of Prop. 30.1.* We define $q(y) = \begin{cases} \iota(x) & \text{if } y = f(x) \\ \infty & \text{if } y \notin f(X). \end{cases}$

To see that $q$ is continuous, let $U \subseteq \hat{X}$ be open. If $\infty \notin U$, then $q^{-1}(U) = f(\iota^{-1}(U))$ is open by the lemma. If $\infty \in U$, then $K = \hat{X} \setminus U$ is compact and thus closed. We have $q^{-1}(K) = f(\iota^{-1}(K))$ is compact and closed in $Y$, so it follows that $q^{-1}(U) = Y \setminus q^{-1}(K)$ is open.

Note that $q$ is automatically a quotient map since it is a closed continuous surjection (it is closed because $Y$ is compact and $\hat{X}$ is Hausdorff). Note also that $q$ is unique because $\hat{X}$ is Hausdorff and $q$ is already specified on the dense subset $f(X) \subseteq Y$. ■

**Remark 30.3.** Note that if we apply the one-point compactification to a (locally compact) metric space $X$, there is no natural metric to put on $X$, so one might ask for a good notion of compactification for metric spaces. Given the result above, this should be related to the idea of a completion of a metric space. See HW8.

The following result is often useful, and it matches more closely what we might have expected the definition of locally compact to resemble.

**Proposition 30.4.** Let $X$ be locally compact and Hausdorff. Let $U$ be an open neighborhood of $x$. Then there is a precompact open set $V$ with $x \in V \subseteq \overline{V} \subseteq U$.

We will need to distinguish between closures in $X$ and closures in $Y$, so we use the notation $\text{cl}_X(A)$ for closure rather than our usual $\overline{A}$. 

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3We will need to distinguish between closures in $X$ and closures in $Y$, so we use the notation $\text{cl}_X(A)$ for closure rather than our usual $\overline{A}$. 

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Proof. We use Prop 29.4. Thus let \( X \hookrightarrow Y \) be the one-point compactification. By definition, \( U \) is still open in \( Y \), so \( K = Y - U \) is closed in \( Y \) and therefore compact. By HW 7.3, we can find disjoint open sets \( V \) and \( W \) in \( Y \) with \( x \in V \) and \( K \subseteq W \). Since \( W \) is open, it follows that \( \overline{V} \) is disjoint from \( W \) and therefore also from \( K \). In other words, \( \overline{V} \) is contained in \( U \).

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**Proposition 30.5.** A space \( X \) is Hausdorff and locally compact if and only if it is homeomorphic to an open subset of a compact Hausdorff space \( Y \).

**Proof.** (\( \Rightarrow \)). We saw that \( X \) is open in the compact Hausdorff space \( Y = \tilde{X} \).

(\( \Leftarrow \)) As a subspace of a Hausdorff space, it is clear that \( X \) is Hausdorff. It remains to show that every point has a compact neighborhood (in \( X \)). Write \( Y_\infty = Y \setminus X \). This is closed in \( Y \) and therefore compact. By Problem 3 from HW7, we can find disjoint open sets \( x \in U \) and \( Y_\infty \subseteq V \) in \( Y \). Then \( K = Y \setminus V \) is the desired compact neighborhood of \( x \) in \( X \).

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**Corollary 30.6.** If \( X \) and \( Y \) are locally compact Hausdorff, then so is \( X \times Y \).

**Corollary 30.7.** Any open or closed subset of a locally compact Hausdorff space is locally compact Hausdorff.

31. Fri, Nov 7

We finally turn to the so-called “separation axioms”.

**Definition 31.1.** A space \( X \) is said to be

- \( T_0 \) if given two distinct points \( x \) and \( y \), there is a neighborhood of one not containing the other
- \( T_1 \) if given two distinct points \( x \) and \( y \), there is a neighborhood of \( x \) not containing \( y \) and vice versa (points are closed)
- \( T_2 \) (Hausdorff) if any two distinct points \( x \) and \( y \) have disjoint neighborhoods
- \( T_3 \) (regular) if points are closed and given a closed subset \( A \) and \( x \notin A \), there are disjoint open sets \( U \) and \( V \) with \( A \subseteq U \) and \( x \in V \)
- \( T_4 \) (normal) if points are closed and given closed disjoint subsets \( A \) and \( B \), there are disjoint open sets \( U \) and \( V \) with \( A \subseteq U \) and \( B \subseteq V \).

Note that \( T_4 \implies T_3 \implies T_2 \implies T_1 \implies T_0 \). But beware that in some literature, the “points are closed” clause is not included in the definition of regular or normal. Without that, we would not be able to deduce \( T_2 \) from \( T_3 \) or \( T_4 \).

We have talked a lot about Hausdorff spaces. The other important separation property is \( T_4 \). We will not really discuss the intermediate notion of regular (or the other variants completely regular, completely normal, etc.)

**Proposition 31.2.** Any compact Hausdorff space is normal.

**Proof.** This was homework problem 7.3.

Later in the course, we will see that this generalizes to locally compact Hausdorff, as long as we add in the assumption that the space is second-countable. Another important class of normal spaces is the collection of metric spaces.

**Proposition 31.3.** If \( X \) is metric, then it is normal.
Proof. Let $X$ be metric and let $A, B \subseteq X$ be closed and disjoint. For every $a \in A$, let $\epsilon_a > 0$ be a number such that $B_{\epsilon_a}(a)$ does not meet $B$ (using that $B$ is closed). Let

$$U_A = \bigcup_{a \in A} B_{\epsilon_a/2}(a).$$

Similarly, we let

$$U_B = \bigcup_{b \in B} B_{\epsilon_b/2}(b).$$

It only remains to show that $U_A$ and $U_B$ must be disjoint. Let $x \in B_{\epsilon_a/2}(a) \subseteq U_A$ and pick any $b \in B$. We have

$$d(a, x) < \frac{1}{2} \epsilon_a < \frac{1}{2} d(a, b)$$

and thus

$$d(x, b) \geq d(a, b) - d(a, x) > d(a, b) - \frac{1}{2} d(a, b) = \frac{1}{2} d(a, b) > \frac{1}{2} \epsilon_b.$$

It follows that $U_A \cap U_B = \emptyset$. 

Unfortunately, the $T_4$ condition alone is not preserved by the constructions we have studied.

Example 31.4. (Images) We will see that $\mathbb{R}$ is normal. But recall the quotient map $q : \mathbb{R} \to \{-1, 0, 1\}$ which sends any number to its sign. This quotient is not Hausdorff and therefore not (regular or) normal.

Example 31.5. (Subspaces) If $J$ is uncountable, then the product $(0, 1)^J$ is not normal (Munkres, example 32.2). This is a subspace of $[0, 1]^J$, which is compact Hausdorff by the Tychonoff theorem and therefore normal. So a subspace of a normal space need not be normal. We also saw in this example that (uncountable) products of normal spaces need not be normal.

Example 31.6. (Products) The lower limit topology $\mathbb{R}_{\ell\ell}$ is normal (Munkres, example 31.2), but $\mathbb{R}_{\ell\ell} \times \mathbb{R}_{\ell\ell}$ is not normal (Munkres, example 31.3). Note that this also gives an example of a Hausdorff space that is not normal.

Ok, so we’ve seen a few examples. So what, why should we care about normal spaces? Look back at the definition for $T_2$, $T_3$, $T_4$. In each case, we need to find certain open sets $U$ and $V$. How would one do this in general? In a metric space, we would build these up by taking unions of balls. In an arbitrary space, we might use a basis. But another way of getting open sets is by pulling back open sets under a continuous map. That is, suppose we have a map $f : X \to [0, 1]$ such that $f \equiv 0$ on $A$ and $f \equiv 1$ on $B$. Then $A \subseteq U := f^{-1}((0, \frac{1}{2}))$ and $B \subseteq V := f^{-1}((\frac{1}{2}, 1])$, and $U \cap V = \emptyset$.

First, note that the definition of normal, by considering the complement of $B$, can be restated as

Lemma 31.7. Let $X$ be normal, and suppose given $A \subseteq U$ with $A$ closed and $U$ open. Then there exists an open set $V$ with

$$A \subseteq V \subseteq \overline{V} \subseteq U.$$

Now we have another very important result.

Theorem 31.8 (Urysohn’s Lemma). Let $X$ be normal and let $A$ and $B$ be disjoint closed subsets. Then there exists a continuous function $f : X \to [0, 1]$ such that $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(1)$.

Sketch of proof. Define $U_1 = X \setminus B$, so that we have $A \subseteq U_1$. Since $X$ is normal, we can find an open $U_0$ with $A \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1$. By induction on the rational numbers $r \in \mathbb{Q} \cap (0, 1)$, we can find for each $r$ an open set $U_r$ with $\overline{U_r} \subseteq U_s$ if $r < s$. We also define $U_r = X$ for $r > 1$. Then define

$$f(x) = \inf \{ r \in \mathbb{Q} \cap (0, 1.001) \mid x \in U_r \}.$$
Now if $x \in A$, then $x \in U_0$, so $f(x) = 0$ as desired. If $x \in B$, then $x \notin U_1$, but $x \in U_r$ for any $r > 1$, so $f(x) = 1$ as desired. It remains to show that $f$ is continuous.

It suffices to show that the preimage under $f$ of the prebasis elements $(-\infty, a)$ and $(a, \infty)$ are open. We have

$$f^{-1}(-\infty, a) = \bigcup_{r \in \mathbb{Q}, r < a} U_r, \quad \text{and} \quad f^{-1}(a, \infty) = \bigcup_{r \in \mathbb{Q}, r > a} X \setminus \overline{U_r}$$

To see the second equality, note that if $f(x) > a$ then for any $a < r < f(x)$, we have $x \notin U_r$. But we can then find $r < s < f(x)$, so that $x \notin U_s \supseteq \overline{U_r} \supseteq U_r$. For more details, see either [Lee, Thm 4.82] or [Munkres, Thm 33.1].

Note that Urysohn’s Lemma becomes an if and only if statement if we either drop the $T_1$-condition from normal or if we explicitly include singletons as possible replacements for $A$ and $B$. 