Locally compact Hausdorff spaces are a very nice class of spaces (almost as good as compact Hausdorff). In fact, any such space is close to a compact Hausdorff space.

Definition 29.1. A **compactification** of a noncompact space X is an embedding $i : X \hookrightarrow Y$, where Y is compact and $i(X)$ is dense.

We will typically work with Hausdorff spaces X, in which case we ask the compactification Y to also be Hausdorff.

Example 29.2. The open interval $(0,1)$ is not compact, but $(0,1) \hookrightarrow [0,1]$ is a compactification. Note that the exponential map $\exp : (0,1) \longrightarrow S^1$ also gives a (different) compactification.

There is often a smallest compactification, given by the following construction.

Definition 29.3. Let X be a space and define $\widehat{X} = X \cup \{\infty\}$, where $U \subseteq \widehat{X}$ is open if either

- $U \subseteq X$ and U is open in X or
- $\infty \in U$ and $\widehat{X} \setminus U \subseteq X$ is compact.

Proposition 29.4. *Suppose that X is Hausdorff and noncompact. Then $\widehat{X}$ is a compactification. If X is locally compact, then $\widehat{X}$ is Hausdorff.*

Proof. We first show that $\widehat{X}$ is a space! It is clear that both $\emptyset$ and $\widehat{X}$ are open.

Suppose that U_1 and U_2 are open. We wish to show that $U_1 \cap U_2$ is open.

- If neither open set contains ∞, this follows since X is a space.
- If $\infty \in U_1$ but $\infty \notin U_2$, then $K_1 = X \setminus U_1$ is compact. Since X is Hausdorff, K_1 is closed in X. Thus $X \setminus K_1 = U_1 \setminus \{\infty\}$ is open in X, and it follows that $U_1 \cap U_2 = (U_1 \setminus \{\infty\}) \cap U_2$ is open.
- If $\infty \in U_1 \cap U_2$, then $K_1 = X \setminus U_1$ and $K_2 = X \setminus U_2$ are compact. It follows that $K_1 \cup K_2$ is compact, so that $U_1 \cap U_2 = X \setminus (K_1 \cup K_2)$ is open.
- Suppose we have a collection U_i of open sets. If none contain ∞, then neither does $\bigcup_i U_i$, and the union is open in X. If $\infty \in U_j$ for some j, then $\infty \in \bigcup_i U_i$ and
$$\widehat{X} \setminus \bigcup_i U_i = \bigcap_i (\widehat{X} \setminus U_i) = \bigcap_i (X \setminus U_i)$$
is a closed subset of the compact set $X \setminus U_j$, so it must be compact.

Next, we show that $\iota : X \longrightarrow \widehat{X}$ is an embedding. Continuity of ι again uses that compact subsets of X are closed. That ι is open follows immediately from the definition of $\widehat{X}$.

To see that $\iota(X)$ is dense in $\widehat{X}$, it suffices to see that $\{\infty\}$ is not open. But this follows from the definition of $\widehat{X}$, since X is not compact.

Finally, we show that $\widehat{X}$ is compact. Let $\mathcal{U}$ be an open cover. Then some $U \in \mathcal{U}$ must contain ∞. The remaining elements of $\mathcal{U}$ must cover $X \setminus U$, which is compact. It follows that we can cover $X \setminus U$ using only finitely many elements, so $\mathcal{U}$ has a finite subcover.

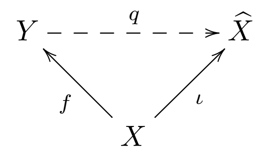
Now suppose that X is locally compact. Let x_1 and x_2 in $\widehat{X}$. If neither is ∞, then we have disjoint neighborhoods in X, and these are still disjoint neighborhoods in $\widehat{X}$. If $x_2 = \infty$, let $x_1 \in U \subseteq K$, where U is open and K is compact. Then U and $V = \widehat{X} \setminus K$ are the desired disjoint neighborhoods . ■

Example 29.5. We saw that S^1 is a one-point compactification of $(0,1) \cong \mathbb{R}$. You will show on your homework that similarly S^n is a one-point compactification of $\mathbb{R}^n$.

Example 29.6. As we have seen, $\mathbb{Q}$ is not locally compact, so we do not expect $\widehat{\mathbb{Q}}$ to be Hausdorff. Indeed, the point ∞ is dense in $\widehat{\mathbb{Q}}$. Because of the topology on $\widehat{\mathbb{Q}}$, this is equivalent to showing that for any open, nonempty subset $U \subseteq \mathbb{Q}$, U is not contained in any compact subset. Since $\mathbb{Q}$ is Hausdorff, if U were contained in a compact subset, then $\overline{U}$ would also be compact. But as we have seen, for any interval $(a, b) \cap \mathbb{Q}$, the closure in $\mathbb{Q}$, which is $[a, b] \cap \mathbb{Q}$, is not compact.

30. Wed, Nov. 5

Next, we show that the situation we observed for compactifications of $(0, 1)$ holds quite generally.



Proposition 30.1. *Let X be locally compact Hausdorff and let $f : X \longrightarrow Y$ be a compactification. Then there is a (unique) quotient map $q : Y \longrightarrow \widehat{X}$ such that $q \circ f = \iota$.*

We will need:

Lemma 30.2. *Let X be locally compact Hausdorff and $f : X \longrightarrow Y$ a compactification. Then f is open.*

Proof. Since f is an emebedding, we can pretend that $X \subseteq Y$ and that f is simply the inclusion. We wish to show that X is open in Y. Thus let $x \in X$. Let U be a precompact neighborhood of x. Thus $K = \mathrm{cl}_X(U)$ is compact[3] and so must be closed in Y (and X) since Y is Hausdorff. By the definition of the subspace topology, we must have $U = V \cap X$ for some open $V \subseteq Y$. Then V is a neighborhood of x in Y, and

$$V = V \cap Y = V \cap \mathrm{cl}_Y(X) \subseteq \mathrm{cl}_Y(V \cap X) = K \subseteq X.$$

■

Proof of Prop. 30.1. We define

$$q(y) = \begin{cases} \iota(x) & \text{if } y = f(x) \\ \infty & \text{if } y \notin f(X). \end{cases}$$

To see that q is continuous, let $U \subseteq \widehat{X}$ be open. If $\infty \notin U$, then $q^{-1}(U) = f(\iota^{-1}(U))$ is open by the lemma. If $\infty \in U$, then $K = \widehat{X} \setminus U$ is compact and thus closed. We have $q^{-1}(K) = f(\iota^{-1}(K))$ is compact and closed in Y, so it follows that $q^{-1}(U) = Y \setminus q^{-1}(K)$ is open.

Note that q is automatically a quotient map since it is a closed continuous surjection (it is closed because Y is compact and $\widehat{X}$ is Hausdorff). Note also that q is unique because $\widehat{X}$ is Hausdorff and q is already specified on the dense subset $f(X) \subseteq Y$. ■

Remark 30.3. Note that if we apply the one-point compactification to a (locally compact) metric space X, there is no natural metric to put on X, so one might ask for a good notion of compactification for metric spaces. Given the result above, this should be related to the idea of a completion of a metric space. See HW8.

The following result is often useful, and it matches more closely what we might have expected the definition of locally compact to resemble.

Proposition 30.4. *Let X be locally compact and Hausdorff. Let U be an open neighborhood of x. Then there is a precompact open set V with*

$$x \in V \subseteq \overline{V} \subseteq U.$$

[3]We will need to distinguish between closures in X and closures in Y, so we use the notation $\mathrm{cl}_X(A)$ for closure rather than our usual $\overline{A}$.

Proof. We use Prop 29.4. Thus let $X \hookrightarrow Y$ be the one-point compactification. By definition, U is still open in Y, so $K = Y - U$ is closed in Y and therefore compact. By HW 7.3, we can find disjoint open sets V and W in Y with $x \in V$ and $K \subseteq W$. Since W is open, it follows that $\overline{V}$ is disjoint from W and therefore also from K. In other words, $\overline{V}$ is contained in U. ■

Proposition 30.5. *A space X is Hausdorff and locally compact if and only if it is homeomorphic to an open subset of a compact Hausdorff space Y.*

Proof. ($\Rightarrow$). We saw that X is open in the compact Hausdorff space $Y = \widehat{X}$.

($\Leftarrow$) As a subspace of a Hausdorff space, it is clear that X is Hausdorff. It remains to show that every point has a compact neighborhood (in X). Write $Y_\infty = Y \setminus X$. This is closed in Y and therefore compact. By Problem 3 from HW7, we can find disjoint open sets $x \in U$ and $Y_\infty \subseteq V$ in Y. Then $K = Y \setminus V$ is the desired compact neighborhood of x in X. ■

Corollary 30.6. *If X and Y are locally compact Hausdorff, then so is $X \times Y$.*

Corollary 30.7. *Any open or closed subset of a locally compact Hausdorff space is locally compact Hausdorff.*

31. Fri, Nov 7

We finally turn to the so-called "separation axioms".

Definition 31.1. A space X is said to be

- T_0 if given two distinct points x and y, there is a neighborhood of one not containing the other
- T_1 if given two distinct points x and y, there is a neighborhood of x not containing y and vice versa (points are closed)
- T_2 (**Hausdorff**) if any two distinct points x and y have disjoint neighborhoods
- T_3 (**regular**) if *points are closed and* given a closed subset A and $x \notin A$, there are disjoint open sets U and V with $A \subseteq U$ and $x \in V$
- T_4 (**normal**) if *points are closed and* given closed disjoint subsets A and B, there are disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$.

Note that $T_4 \implies T_3 \implies T_2 \implies T_1 \implies T_0$. But beware that in some literature, the "points are closed" clause is not included in the definition of regular or normal. Without that, we would not be able to deduce T_2 from T_3 or T_4.

We have talked a lot about Hausdorff spaces. The other important separation property is T_4. We will not really discuss the intermediate notion of regular (or the other variants completely regular, completely normal, etc.)

Proposition 31.2. *Any compact Hausdorff space is normal.*

Proof. This was homework problem 7.3. ■

Later in the course, we will see that this generalizes to locally compact Hausdorff, as long as we add in the assumption that the space is second-countable. Another important class of normal spaces is the collection of metric spaces.

Proposition 31.3. *If X is metric, then it is normal.*

Proof. Let X be metric and let $A, B \subseteq X$ be closed and disjoint. For every $a \in A$, let $\epsilon_a > 0$ be a number such that $B_{\epsilon_a}(a)$ does not meet B (using that B is closed). Let

$$U_A = \bigcup_{a \in A} B_{\epsilon_a/2}(a).$$

Similarly, we let

$$U_B = \bigcup_{b \in B} B_{\epsilon_b/2}(b).$$

It only remains to show that U_A and U_B must be disjoint. Let $x \in B_{\epsilon_a/2}(a) \subseteq U_A$ and pick any $b \in B$. We have

$$d(a,x) < \frac{1}{2}\epsilon_a < \frac{1}{2}d(a,b)$$

and thus

$$d(x,b) \geq d(a,b) - d(a,x) > d(a,b) - \frac{1}{2}d(a,b) = \frac{1}{2}d(a,b) > \frac{1}{2}\epsilon_b.$$

It follows that $U_A \cap U_B = \emptyset$. ■

Unfortunately, the T_4 condition alone is not preserved by the constructions we have studied.

Example 31.4. (Images) We will see that $\mathbb{R}$ is normal. But recall the quotient map $q : \mathbb{R} \longrightarrow \{-1, 0, 1\}$ which sends any number to its sign. This quotient is not Hausdorff and therefore not (regular or) normal.

Example 31.5. (Subspaces) If J is uncountable, then the product $(0,1)^J$ is *not* normal (Munkres, example 32.2). This is a subspace of $[0,1]^J$, which is compact Hausdorff by the Tychonoff theorem and therefore normal. So a subspace of a normal space need not be normal. We also saw in this example that (uncountable) products of normal spaces need not be normal.

Example 31.6. (Products) The lower limit topology $\mathbb{R}_{\ell\ell}$ is normal (Munkres, example 31.2), but $\mathbb{R}_{\ell\ell} \times \mathbb{R}_{\ell\ell}$ is *not* normal (Munkres, example 31.3). Note that this also gives an example of a Hausdorff space that is not normal.

Ok, so we've seen a few examples. So what, why should we care about normal spaces? Look back at the definition for T_2, T_3, T_4. In each case, we need to find certain open sets U and V. How would one do this in general? In a metric space, we would build these up by taking unions of balls. In an arbitrary space, we might use a basis. But another way of getting open sets is by pulling back open sets under a continuous map. That is, suppose we have a map $f : X \longrightarrow [0,1]$ such that $f \equiv 0$ on A and $f \equiv 1$ on B. Then $A \subseteq U := f^{-1}([0, \frac{1}{2}))$ and $B \subseteq V := f^{-1}((\frac{1}{2}, 1])$, and $U \cap V =$.

First, note that the definition of normal, by considering the complement of B, can be restated as

Lemma 31.7. *Let X be normal, and suppose given $A \subseteq U$ with A closed and U open. Then there exists an open set V with*

$$A \subseteq V \subseteq \overline{V} \subseteq U.$$

Now we have another very important result.

Theorem 31.8 (Urysohn's Lemma)**.** *Let X be normal and let A and B be disjoint closed subsets. Then there exists a continuous function $f : X \longrightarrow [0,1]$ such that $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(1)$.*

Sketch of proof. Define $U_1 = X \setminus B$, so that we have $A \subseteq U_1$. Since X is normal, we can find an open U_0 with $A \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1$. By induction on the rational numbers $r \in \mathbb{Q} \cap (0,1)$, we can find for each r an open set U_r with $\overline{U_r} \subset U_s$ if $r < s$. We also define $U_r = X$ for $r > 1$. Then define

$$f(x) = \inf\{r \in \mathbb{Q} \cap [0, 1.001) \mid x \in U_r\}.$$

Now if $x \in A$, then $x \in U_0$, so $f(x) = 0$ as desired. If $x \in B$, then $x \notin U_1$, but $x \in U_r$ for any $r > 1$, so $f(x) = 1$ as desired. It remains to show that f is continuous.

It suffices to show that the preimage under f of the prebasis elements $(-\infty, a)$ and (a, ∞) are open. We have

$$f^{-1}(-\infty, a) = \bigcup_{\substack{r \in \mathbb{Q} \\ r < a}} U_r, \qquad \text{and} \qquad f^{-1}(a, \infty) = \bigcup_{\substack{r \in Q \\ r > a}} X \setminus \overline{U_r}$$

To see the second equality, note that if $f(x) > a$ then for any $a < r < f(x)$, we have $x \notin U_r$. But we can then find $r < s < f(x)$, so that $x \notin U_s \supseteq \overline{U_r} \supseteq U_r$. For more details, see either [Lee, Thm 4.82] or [Munkres, Thm 33.1]. ■

Note that Urysohn's Lemma becomes an if and only if statement if we either drop the T_1-condition from normal or if we explicitly include singletons as possible replacements for A and B.