Last time, we saw that a space is normal if and only if any two closed sets can be separated by a continuous function (modulo the $T_1$ condition). Here is another important application of normal spaces.

**Theorem 32.1** (Tietze extension theorem). Suppose $X$ is normal and $A \subseteq X$ is closed. Then any continuous function $f : A \rightarrow [0,1]$ can be extended to a continuous function $\tilde{f} : X \rightarrow [0,1]$.

Again, this becomes an if and only if if we drop the $T_1$-condition from normal.

It is also easy to see that the result fails if we drop the hypothesis that $A$ is closed. Consider $X = S^1$ and $A$ is the complement of a point. Then we know that $A \cong (0,1)$, but this homeomorphism cannot extend to a map $S^1 \rightarrow (0,1)$.

**Sketch of proof.** It is more convenient for the purpose of the proof to work with the interval $[-1,1]$ rather than $[0,1]$. Thus suppose $f : A \rightarrow [-1,1]$ is continuous. Then $A_1 = f^{-1}([-1,-1/3])$ and $B_1 = f^{-1}([1/3,1])$ are closed, disjoint subsets of $A$ and therefore also of $X$. Since $X$ is normal, we have a Urysohn function $g_1 : X \rightarrow [-1/3,1/3]$ which separates $A_1$ and $B_1$. It is simple to check that $|f(a) - g_1(a)| \leq 2/3$ for all $a \in A$. In other words, we have a map $f_1 = f - g_1 : A \rightarrow [-2/3,2/3]$.

Define $A_2 = f_1^{-1}([-2/3,-2/9])$ and $B_2 = f_1^{-1}([2/9,2/3])$. We get a Urysohn function $g_2 : X \rightarrow [-2/9,2/3]$ which separates $A_2$ and $B_2$. Then the difference $f_2 = f - g_1 - g_2$ maps to $[-4/9,4/9]$.

We continue in this way, and in the end, we get a sequence of functions $(g_n)$ defined on $X$, and we define $g = \sum_n g_n$. By construction, this agrees with $f$ on $A$ (the difference will be less than $(2/3)^n$ for all $n$). The work remains in showing that the series defining $g$ converges (compare to a geometric series) and that the resulting $g$ is continuous (show that the series converges uniformly). See [Munkres, Thm 35.1] for more details.

**Theorem 32.2** (Stone-Čech compactification). Suppose $X$ is normal. There exists a “universal” compactification $\iota : X \rightarrow Y$ of $X$, such that if $j : X \rightarrow Z$ is any map to a compact Hausdorff space (for example a compactification), there is a unique map $q : Y \rightarrow Z$ with $q \circ \iota = j$.

**Proof.** Given the space $X$, let $\mathcal{F} = \{\text{cts } f : X \rightarrow [0,1]\}$. Define $\iota : X \rightarrow [0,1]^\mathcal{F}$ by $\iota(x) = f(x)$. This is continuous because each coordinate function is given by some $f \in \mathcal{F}$. The infinite cube is compact Hausdorff, and we let $Y = \overline{\iota(X)}$. It remains to show that $\iota$ is an embedding and also to demonstrate the universal property.

First, $\iota$ is injective by Urysohn’s lemma: given distinct points $x$ and $y$ in $X$, there is a Urysohn function separating $x$ and $y$, so $\iota(x) \neq \iota(y)$.

Now suppose that $U \subseteq X$ is open. We wish to show that $\iota(U)$ is open in $\iota(X)$. Pick $x_0 \in U$. Again by Urysohn’s lemma, we have a function $g : X \rightarrow [0,1]$ with $g(x_0) = 0$ and $g \equiv 1$ outside of $U$. Let $B = \{\iota(x) \in \iota(X) \mid g(x) \neq 1\} = \iota(X) \cap p^{-1}_g([0,1])$.

Certainly $\iota(x_0) \in B$. Finally, $B \subset \iota(U)$ since if $\iota(x) \in B$, then $g(x) \neq 1$. But $g \equiv 1$ outside of $U$, so $x$ must be in $U$. 

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For the universal property, suppose that \( j : X \to Z \) is a map to a compact Hausdorff space. Then \( Z \) is also normal, and the argument above shows that it embeds inside some large cube \([0,1]^K\). For each \( k : Z \to [0,1] \) in \( K \), we thus get a coordinate map \( i_k = p_k \circ j : X \to [0,1] \), and it is clear how to extend this to get a map \( q_k : Y \to [0,1] \); just take \( q_k \) to be the projection map \( p_{i_k} \) onto the factor labelled by the map \( i_k \). Piecing these together gives a map \( q : Y \to [0,1]^K \), but it restricts to the map \( j \) on the subset \( X \). Since \( j \) has image in the closed subset \( Z \), it follows that \( q(Y) \subseteq Z \) since \( q \) is continuous and \( i(X) \) is dense in \( Y \). Note that \( q \) is the unique extension of \( j \) to \( Y \) since \( Z \) is Hausdorff and \( i(X) \) is dense in \( Y \).

**Corollary 33.1.** Suppose that \( X \) is normal, and that \( X \hookrightarrow Z \) is any compactification. Then \( Z \) is a quotient of the Stone-Čech compactification \( Y \) of \( X \).

**Proof.** According to the Theorem 32.2, we have a continuous map \( q : Y \to Z \) whose restriction to \( X \) is the given map \( j : X \to Z \). The map \( q \) is closed since \( Y \) is compact and \( Z \) is Hausdorff. Also, \( j(X) \) is dense in \( Z \), and \( j(X) = q(\iota(X)) \subseteq q(Y) \) so \( q(Y) = Z \). In other words, \( q \) is closed, continuous, and surjective, therefore it is a quotient map.

The Stone-Čech compactification has consequences for *metrizability* of a space. Consider first the case that the index set \( J \) is countable.

**Proposition 33.2.** Let \( Y \) be a metric space, and let \( \overline{d} : Y \times Y \to \mathbb{R} \) be the associated truncated metric. Then the formula

\[
D(y, z) = \sup \left\{ \frac{\overline{d}(y_n, z_n)}{n} \right\}
\]

defines a metric on \( Y^\mathbb{N} \), and the induced topology is the product topology.

**Proof.** We leave as an exercise the verification that this is a metric. We check the statement about the topology. For each \( n \), let \( p_n : Y^\mathbb{N} \to Y \) be evaluation in the \( n \)th place. This is continuous, as given \( y \in Y^\mathbb{N} \) and \( \epsilon > 0 \), we let \( \delta = \epsilon/n \). Then if \( D(y, z) < \delta \), it follows that

\[
d(y_n, z_n) = n \frac{d(y_n, z_n)}{n} \leq nD(y, z) < n\delta = \epsilon.
\]

By the universal property of the product, we get a continuous bijection \( p : Y^\mathbb{N} \to \prod_{\mathbb{N}} Y \).

It remains to show that \( p \) is open. Thus let \( B \subseteq Y^\mathbb{N} \) be an open ball, and let \( y \in p(B) = B \). We want to find a basis element \( U \) in the product topology with \( y \in U \subseteq B \). For convenience, we replace \( B \) by \( B_\epsilon(y) \) for small enough \( \epsilon \). Take \( N \) large such that \( 1/N < \epsilon \). Then define

\[
U = \bigcap_{i=1}^N p_i^{-1}(B_\epsilon(y_i)).
\]

Let \( z \in Y^\mathbb{N} \). Recall that we have truncated our metric on \( Y \) at \( 1 \). Thus if \( n > N \), we have that \( \overline{d}(y_n, z_n)/n \leq 1/n \leq 1/N < \epsilon \). It follows that for any \( z \in U \), we have \( z \in B_\epsilon(x) \) as desired.

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On the other hand, if \( J \) is uncountable, then \([0,1]^J\) need not be metric, as the following example shows.

**Example 34.1.** The sequence lemma fails in \( \mathbb{R}^\mathbb{R} \). Let \( A \subseteq \mathbb{R}^\mathbb{R} \) be the subset consisting of functions that zero at all but finitely many points. Let \( g \) be the constant function at \( 1 \). Then \( g \in \overline{A} \), since if

\[
U = \bigcap_{x_1, \ldots, x_k} p_{x_1}^{-1}(a_i, b_i)
\]
is a neighborhood of \( g \), then the function

\[
f(x) = \begin{cases} 
1 & x \in \{x_1, \ldots, x_k\} \\
0 & \text{else}
\end{cases}
\]

is in \( U \cap A \). But no sequence in \( A \) can converge to \( g \) (recall that convergence in the product topology means pointwise convergence). For suppose \( f_n \) is a sequence in \( A \). For each \( n \), let \( Z_n = \text{supp}(f_n) \) (the support is the set where \( f_n \) is nonzero). Then the set

\[
Z = \bigcup_n Z_n
\]

is countable, and on the complement of \( Z \), all \( f_n \)'s are zero. So it follows that the same must be true for any limit of \( f_n \). Thus the \( f_n \) cannot converge to \( g \).

This finally leads to a characterization of those topological spaces which come from metric spaces.

**Theorem 34.2.** If \( X \) is normal and second countable, then it is metrizable.

**Proof.** Since \( X \) is normal, we can embed \( X \) as above inside a cube \( [0,1]^J \) for some \( J \). Above, we took \( J \) to be the collection of all functions \( X \rightarrow [0,1] \).

To get a countable indexing set \( J \), start with a countable basis \( B = \{B_n\} \) for \( X \). For each pair of indices \( n, m \) for which \( \overline{B}_n \subset B_m \), the Urysohn lemma gives us a function \( g_{n,m} \) vanishing on \( B_n \) and equal to 1 outside \( B_m \). We take \( J = \{g_{n,m}\} \). Going back to the proof of the Stone-Čech-compactification, we needed, for any \( x_0 \in X \) and \( x_0 \in U \), to be able to find a function vanishing at \( x_0 \) but equal to 1 outside of \( U \).

Take a basis element \( B_m \) satisfying \( x_0 \in B_m \subset U \). Since \( X \) is normal, we can find an open set \( V \) with \( x_0 \in V \subset \overline{V} \subset B_m \). Find a \( B_n \) inside of \( V \), and we are now done: namely, the function \( g_{n,m} \) is what we were after.

We now come back to a result that we previously put off.

**Theorem 34.3.** Suppose \( X \) is locally compact, Hausdorff, and second-countable. Then \( X \) is normal.

**Proof.** Given closed, disjoint subsets \( A \) and \( B \), we want to separate them using disjoint open sets.

Consider first the case where \( A = \{a\} \) is a point. Writing \( V = X \setminus B \), we have \( a \in V \), and we wish to find \( U \) with \( a \in U \subset \overline{U} \subset V \). Since \( X \) is locally compact, Hausdorff, we can consider the one-point compactification \( \hat{X} \). But now we have \( a \in V \subset \hat{X} \), and \( \hat{X} \) is compact Hausdorff and therefore normal. So we get the desired \( U \). Note that the same argument does not work for a general \( A \), since we would not know that \( A \) is closed in \( \hat{X} \) (unless \( A \) is compact). We have proved that \( X \) is regular \((T_3)\).

Now let \( A \) and \( B \) be general closed, disjoint subsets. For each \( a \in A \), we can find a basis element \( U_a \) with \( a \in U_a \subset \overline{U}_a \subset X \setminus B \). Since our basis is countable, we can enumerate all such \( U_a \)'s to get a countable cover \( \{U_n\} \) of \( A \) which is disjoint from \( B \). Similarly, we get a countable cover \( \{V_n\} \) of \( B \) which is disjoint from \( A \). But the \( U_n \)'s need not be disjoint from the \( V_k \)'s so we need to fix this.

Define new covers of \( A \) and \( B \), respectively, as follows. For each \( n \), define

\[
\tilde{U}_n = U_n \setminus \bigcup_{j=1}^n V_j \quad \text{and} \quad \tilde{V}_n = V_n \setminus \bigcup_{j=1}^n U_j
\]

The \( \tilde{U}_n \)'s still cover \( A \) because we have removed the \( V_j \), which were all disjoint from \( A \). Similarly, the \( \tilde{V}_n \) cover \( B \). Moreover, \( \tilde{U}_n \) is disjoint from \( \tilde{V}_j \) because, assuming WLOG that \( n < j \), the closure of \( U_n \) has been removed from \( V_j \) in the formation of \( V_j \).

Combining the previous results gives
Corollary 34.4. Suppose $X$ is locally compact, Hausdorff, and second-countable. Then $X$ is metrizable.