35. Monday, Nov. 17

Exam day.

36. Wednesday, Nov. 19

We finally arrive at one of the most important definitions of the course.

Definition 36.1. A (topological) **n-manifold** M is a Hausdorff, second-countable space such that each point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

Example 36.2.

(1) $\mathbb{R}^n$ and any open subset is obviously an n-manifold

(2) S^1 is a 1-manifold. More generally, S^n is an n-manifold. Indeed, we have shown that if you remove a point from S^n, the resulting space is homeomorphic to $\mathbb{R}^n$.

(3) T^n, the n-torus, is an n-manifold. In general, if M is an m-manifold and N is an n-manifold, then $M \times N$ is an $(m+n)$-manifold.

(4) $\mathbb{RP}^n$ is an n-manifold. There is a standard covering of $\mathbb{RP}^n$ by open sets as follows. Recall that $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^\times$. For each $1 \le i \le n+1$, let $V_i \subseteq \mathbb{R}^{n+1} \setminus \{0\}$ be the complement of the hyperplane $x_i = 0$. This is an open, saturated set, and so its image $U_i = V_i/\mathbb{R}^\times \subseteq \mathbb{RP}^n$ is open. The V_i's cover $\mathbb{R}^{n+1} \setminus \{0\}$, so the U_i's cover $\mathbb{RP}^n$. We leave the rest of the details as an exercise.

(5) $\mathbb{CP}^n$ is a $2n$-manifold. This is similar to the description given above.

(6) $O(n)$ is a $\frac{n(n-1)}{2}$-manifold. Since it is also a topological group, this makes it a *Lie group*. The standard way to see that this is a manifold is to realize the orthogonal group as the preimage of the identity matrix under the transformation $M_n(R) \longrightarrow M_n(R)$ that sends A to A^TA. This map lands in the subspace $S_n(R)$ of symmetric $n \times n$ matrices. This space can be identified with $\mathbb{R}^{n(n+1)/2}$.

 Now the $n \times n$ identity matrix is an element of S_n, and an important result in differential topology (Sard's theorem) that says that if a certain derivative map is surjective, then the preimage of the submanifold $\{I_n\}$ will be a submanifold of $M_n(\mathbb{R})$ of the same "codimension". in this case, the relevant derivative is the matrix of partial derivatives of $A \mapsto A^TA$, writen in a suitable basis. It follows that

$$\dim O(n) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

 The dimension statement can also be seen directly as follows. If A is an orthogonal matrix, its first column is just a point of S^{n-1}. Then its second column is a point on the sphere orthogonal to the first column, so it lives in an "equator", meaning a sphere of dimension one less. Continuing in this way, we see that the "degree of freedom" for specifying a point of $O(n)$ is $(n-1)+(n-2)+\cdots+1 = \frac{n(n-1)}{2}$.

(7) $\mathrm{Gr}_{k,n}(\mathbb{R})$ is a $k(n-k)$-manifold. One way to see this is to use the homeomorphism

$$\mathrm{Gr}_{k,n}(\mathbb{R}) \cong O(n)/\big(O(k) \times O(n-k)\big)$$

from Example 18.1. We get

$$\begin{aligned}\dim \mathrm{Gr}_{n,k}(\mathbb{R}) &= \dim O(n) - \big(\dim O(k) + \dim O(n-k)\big)\\ &= \sum_{j=1}^{n-1} j - \left(\sum_{j=1}^{k-1} j + \sum_{\ell=1}^{n-k-1} \ell\right) = \sum_{j=k}^{n-1} j - \sum_{\ell=1}^{n-k-1} \ell\\ &= \sum_{\ell=0}^{n-k-1} k+\ell - \sum_{\ell=0}^{n-k-1} \ell = \sum_{\ell=0}^{n-k-1} k = k(n-k)\end{aligned}$$

Here are some nonexamples of manifolds.

Example 36.3.
(1) The union of the coordinate axes in $\mathbb{R}^2$. Every point has a neighborhood like $\mathbb{R}^1$ except for the origin.
(2) A discrete uncountable set is not second countable.
(3) A 0-manifold is discrete, so $\mathbb{Q}$ is not a 0-manifold.
(4) Glue together two copies of $\mathbb{R}$ by identifying any nonzero x in one copy with the point x in the other. This is second-countable and looks locally like $\mathbb{R}^1$, but it is not Hausdorff.

37. Fri, Nov. 21

Proposition 37.1. *Any manifold is normal.*

Proof. This follows form Theorem 34.3. To see that a manifold M is locally compact, consider a point $x \in M$. Then x has a Euclidean neighborhood $x \in U \subseteq M$. U is homeomorphic to an open subset V of $\mathbb{R}^n$, so we can find a compact neighborhood K of x in V (think of a closed ball in $\mathbb{R}^n$). Under the homeomorphism, K corresponds to a compact neighborhood of x in U. ∎

It also follows similarly that any manifold is metrizable, but we can do better. It is convenient to introduce the following term.

Recall that the **support** of a continuous function $f : X \longrightarrow \mathbb{R}$ is $\mathrm{supp}(f) = \overline{f^{-1}(\mathbb{R} \setminus \{0\})}$.

Definition 37.2. Let $\mathcal{U} = \{U_1, \dots U_n\}$ be a finite cover of X. A **partition of unity** subordinate to $\mathcal{U}$ is a collection $\varphi_j \longrightarrow [0,1]$ of continuous functions such that

(1) $\mathrm{supp}(\varphi_\alpha) \subseteq U_\alpha$
(2) we have $\sum_j \varphi_j = 1$.

Theorem 37.3. *Let $\mathcal{U} = \{U_1, \dots, U_n\}$ be a finite covering of the normal space X. Then there is a partition of unity subordinate to $\mathcal{U}$.*

Lemma 37.4. *Let $\mathcal{U} = \{U_1, \dots, U_n\}$ be a finite covering of the normal space X. Then there is a finite cover $\mathcal{V} = \{V_1, \dots, V_n\}$ such that $V_i \subseteq \overline{V_i} \subseteq U_i$ for all i.*

Proof. We give only the argument in the case $n = 2$. Let $A = X \setminus U_2$. Then $A \subseteq U_1$, so we can find an open V_1 with $A \subseteq V_1 \subseteq \overline{V_1} \subseteq U_1$. Now $\{V_1, U_2\}$ is an open cover of X. In the same way, we replace U_2 be a V_2 with $X \setminus V_1 \subseteq V_2 \subseteq \overline{V_2} \subseteq U_2$. ∎

Proof of Theorem 37.3. We use the lemma twice, to get finite covers $\{V_1, \dots, V_n\}$ and $\{W_1, \dots, W_n\}$ with

$$W_i \subseteq \overline{W_i} \subseteq V_i \subseteq \overline{V_i} \subseteq U_i$$

for all i. For each i we have a Urysohn function $g_i : X \longrightarrow [0,1]$ with $g_i \equiv 1$ on $\overline{W_i}$ and vanishing outside of V_i. Note that this implies that $\mathrm{supp}(g_i) \subseteq \overline{V_i} \subseteq U_i$. Since the W_i cover X, it follows that if we define $G = \sum_i g_i$, then $G(x) \geq 1$ for all x. Thus $\varphi_i = g_i/G$ is a continuous function taking values in $[0, 1]$, and we get

$$\sum_i \varphi_i = \sum_i \frac{g_i}{G} = \frac{\sum_i g_i}{\sum_i g_i} = 1.$$

■

Theorem 37.5. *Any manifold M^n admits an embedding into some Euclidean space $\mathbb{R}^N$.*

Proof. The theorem is true as stated, but we only prove it in the case of a compact manifold. Note that in this case, since M is compact and $\mathbb{R}^N$ is Hausdorff, it is enough to find a continuous injection of M into some $\mathbb{R}^N$.

Since M is a manifold, it has an open cover by sets that are homeomorphic to $\mathbb{R}^n$. Since it is compact, there is a finite subcover $\{U_1, \ldots, U_k\}$. By Theorem 37.3, there is a partition of unity $\{\varphi_1, \ldots, \varphi_k\}$ subordinate to this cover. For each i, let $f_i : U_i \xrightarrow{\cong} \mathbb{R}^n$ be a homeomorphism. We can then piece these together as follows: for each $i = 1, \ldots, k$, define $g_i : M \longrightarrow \mathbb{R}^n$ by

$$g_i(x) = \begin{cases} f_i(x)\varphi_i(x) & x \in U_i \\ \mathbf{0} & x \in X \setminus \operatorname{supp}(\varphi_i) \end{cases}.$$

Note that g_i is continuous by the glueing lemma, since $\operatorname{supp}(\varphi_i)$ is closed. Then the k functions g_i together give a continuous function $g : M \longrightarrow \mathbb{R}^{nk}$. Unfortunately, this need not be injective, since if $f_i(x) = \mathbf{0}$ and x does not lie in any other U_j, it follows that $g(x) = \mathbf{0}$. Since there can be more than one such x, we cannot conclude that g is injective.

One way to fix this would be to stick on the functions φ_i, in order to separate out points lying in different U_i's. Define $G = (g_1, \ldots, g_k, \varphi_1, \ldots, \varphi_k) : M \longrightarrow \mathbb{R}^{nk+k}$. But now G is injective, since if $G(x) = G(x')$ and we pick i so that $\varphi_i(x) = \varphi_i(x') > 0$, then this means that x and x' both lie in U_i. But then $g_i(x) = g_i(x')$ so $f_i(x) = f_i(x')$. Since f_i is a homeomorphism, it follows that $x = x'$. ■

In fact, one can do better. Munkres shows (Cor. 50.8) that every compact n-manifold embeds inside $\mathbb{R}^{2n+1}$.

Last time, we discussed some of the nice properties of manifolds. Here is one more we did not get to.

Proposition 38.1. *Any manifold is locally path-connected.*

This follows immediately since a manifold is locally Euclidean.

Another related concept is that of paracompactness. This is especially important in the theory of manifolds and vector bundles. We make a couple of preliminary definitions first.

Definition 38.2. If $\mathcal{U}$ and $\mathcal{W}$ are collections of subsets of X, we say that $\mathcal{W}$ is a **refinement** of $\mathcal{U}$ if every $W \in \mathcal{W}$ is a subset of some $U \in \mathcal{U}$.

Definition 38.3. An open cover $\mathcal{U}$ of X is said to be **locally finite** if every $x \in X$ has a neighborhood meeting only finitely many elements of the cover.

For example, the covering $\{(n, n+2) \mid n \in \mathbb{Z}\}$ of $\mathbb{R}$ is locally finite.

Definition 38.4. A space X is said to be **paracompact** if every open cover has a locally finite refinement.

From the definition, it is clear that compact implies paracompact. But this really is a generalization, as the next example shows.

Proposition 38.5. *The space $\mathbb{R}$ is paracompact.*

Proof. Let $\mathcal{U}$ be an open cover of $\mathbb{R}$. For each $n \geq 0$, let $A_n = \pm[n, n+1]$ and $W_n = \pm(n - \frac{1}{2}, n + \frac{3}{2})$. Then $A_n \subset W_n$, A_n is compact and W_n is open. (We take $W_0 = (-\frac{3}{2}, \frac{3}{2})$.) Fix an n. For each $x \in A_n$, pick a $U_x \in \mathcal{U}$ with $x \in U_x$, and let $V_x = U_x \cap W_n$. The V_x's give an open cover of A_n, and so there is a finite collection $\mathcal{V}_n$ of V_x's that will cover A_n. Then $\mathcal{V} = \bigcup_n \mathcal{V}_n$ gives a locally finite refinement of $\mathcal{U}$. (Note that only W_{n-1}, W_n, and W_{n+1} meet the subset A_n). ∎

This argument adapts easily to show that $\mathbb{R}^n$ is paracompact. In fact, something more general is true.

Lemma 38.6. *Any open cover of a second countable space has a countable subcover.*

Proof. Given a countable basis $\mathcal{B}$ and an open cover $\mathcal{U}$, we first replace the basis by the countable subset $\mathcal{B}'$ consisting of those basis elements that are entirely contained in some open set from the cover (this is a basis too, but we don't need that). For each $B \in \mathcal{B}'$, pick some $U_B \in \mathcal{U}$ containing B, and let $\mathcal{U}' \subseteq \mathcal{U}$ be the (countable) collection of such U_B. It only remains to observe that $\mathcal{U}'$ is still a cover, because

$$\bigcup_{\mathcal{U}'} U_B \supset \bigcup_{\mathcal{B}'} B = X.$$

∎

Proposition 38.7. *Every second countable, locally compact Hausdorff space is paracompact.*

The proof strategy is the same. The assumptions give you a cover (basis) by precompact sets and thus a countable cover by precompact sets. You use this to manufacture a countable collection of compact sets A_n and open sets W_n that cover X as above. The rest of the proof is the same.

Note that of the assumptions in the proposition, locally compact and Hausdorff are both *local* properties, whereas second countable is a global property. As we will see, paracompactness (and therefore the assumptions in this proposition) is enough to guarantee the existence of some nice functions on a space.

Corollary 38.8. *Any manifold is paracompact.*

Theorem 38.9 (Munkres, Theorem 41.4)**.** *If X is metric, then it is paracompact.*

Next, we show that paracompact and Hausdorff implies normal. First, we need a lemma.

Lemma 38.10. *If $\{A\}$ is a locally finite collection of subsets of X, then*

$$\overline{\bigcup A} = \bigcup \overline{A}.$$

Proof. We have already shown before that the inclusion $\supset$ holds generally. The other implication follows from the neighborhood criterion for the closure. Let $x \in \overline{\bigcup A}$. Then we can find a neighborhood U of x meeting only $A_1, \ldots, A_n$. Then $x \in \overline{\bigcup_{i=1}^n A_i}$ since else there would be a neighborhood V of x missing the A_i's. Then $U \cap V$ would be a neighborhood missing $\bigcup A$. But $\overline{\bigcup_{i=1}^n A_i} = \bigcup_{i=1}^n \overline{A_i}$, so we are done. ∎

Theorem 38.11 (Lee, Theorem 4.81)**.** *If X is paracompact and Hausdorff, then it is normal.*

Proof. We first use the Hausdorff assumption to show that X is regular. A similar argument can then be made, using regularity, to show normality.

Thus let A be closed and $b \notin A$. We wish to find disjoint open sets $A \subseteq U$ and $b \in V$. For every $a \in A$, we can find disjoint open neighborhoods U_a of a and V_a of b. Then $\{U_a\} \cup \{X \setminus A\}$ is an open cover, so there is a locally finite subcover $\mathcal{V}$. Take $\mathcal{W} \subseteq \mathcal{V}$ to be the $W \in \mathcal{V}$ such that $W \subseteq U_a$ for some a. Then $\mathcal{W}$ is still locally finite.

We take $U = \bigcup_{W \in \mathcal{W}} W$ and $V = X \setminus \overline{U}$. We know $b \in V$ since $\overline{U} = \bigcup \overline{W}$, and $b \notin \overline{W}$ since $W \subseteq U_a$ and b has a neighborhood (V_a) disjoint from U_a. ∎

Definition 38.12. Let $\mathcal{U} = \{U_\alpha\}$ be a cover of X. A **partition of unity** subordinate to $\mathcal{U}$ is a collection $\varphi_\alpha : X \longrightarrow [0,1]$ of continuous functions such that

(1) $\operatorname{supp}(\varphi_\alpha) \subseteq U_\alpha$
(2) the collection $\operatorname{supp}(\varphi_\alpha)$ is locally finite
(3) we have $\sum_\alpha \varphi_\alpha = 1$. Note that, when evaluated at some $x \in X$, this sum is always finite by the local finite assumption (2).

Theorem 38.13. *Let X be paracompact Hausdorff, and let $\mathcal{U} = \{U_\alpha\}$ be an open cover. Then there exists a partition of unity subordinate to $\mathcal{U}$.*

Lemma 38.14 (Lee, 4.84)**.** *There exists a locally finite refinement $\{V_\alpha\}$ of $\{U_\alpha\}$ with $\overline{V_\alpha} \subseteq U_\alpha$.*

Proof of Theorem. We apply the lemma twice to get locally finite covers $\{V_\alpha\}$ and $\{W_\alpha\}$ with $\overline{W_\alpha} \subseteq V_\alpha \subseteq \overline{V_\alpha} \subseteq U_\alpha$. For each α, we use Urysohn's lemma to get $f_\alpha : X \longrightarrow [0,1]$ with $f_\alpha \equiv 1$ on $\overline{W_\alpha}$ and $\operatorname{supp}(f_\alpha) \subseteq \overline{V_\alpha} \subseteq U_\alpha$. Since $\{V_\alpha\}$ is locally finite, we can define $f : X \longrightarrow [0,1]$ by $f = \sum_\alpha f_\alpha$. Locally around some $x \in X$, the function f is a finite sum of f_α's, and so is continuous. It only remains to normalize our f_α's. Note that at any $x \in X$, we can find an α for which $x \in W_\alpha$, and so $f(x) \geq f_\alpha(x) = 1$. Thus it makes sense to define $\varphi_\alpha : X \longrightarrow [0,1]$ by

$$\varphi_\alpha(x) = \frac{f_\alpha(x)}{f(x)}.$$

We have $\operatorname{supp}(\varphi_\alpha) = \operatorname{supp}(f_\alpha)$, and so the φ_α give a partition of unity. ∎

$$\begin{tikzcd}
\text{metric} \arrow[r, Rightarrow] & \text{paracompact \& Hausdorff} \arrow[r, Rightarrow] & \text{normal} \\
& \begin{array}{c}\text{locally compact, Hausdorff,}\\ \text{\& second-countable}\end{array} \arrow[u, Rightarrow] \arrow[r, Rightarrow] & \begin{array}{c}\text{normal \&}\\ \text{second-countable}\end{array} \arrow[u, Rightarrow] \arrow[ull, Rightarrow, bend left]
\end{tikzcd}$$