Exam day.

36. Wednesday, Nov. 19

We finally arrive at one of the most important definitions of the course.

**Definition 36.1.** A (topological) $n$-manifold $M$ is a Hausdorff, second-countable space such that each point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

**Example 36.2.**

1. $\mathbb{R}^n$ and any open subset is obviously an $n$-manifold.

2. $S^1$ is a 1-manifold. More generally, $S^n$ is an $n$-manifold. Indeed, we have shown that if you remove a point from $S^n$, the resulting space is homeomorphic to $\mathbb{R}^n$.

3. $T^n$, the $n$-torus, is an $n$-manifold. In general, if $M$ is an $m$-manifold and $N$ is an $n$-manifold, then $M \times N$ is an $(m+n)$-manifold.

4. $\mathbb{RP}^n$ is an $n$-manifold. There is a standard covering of $\mathbb{RP}^n$ by open sets as follows. Recall that $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^\times$. For each $1 \leq i \leq n+1$, let $V_i \subseteq \mathbb{R}^{n+1} \setminus \{0\}$ be the complement of the hyperplane $x_i = 0$. This is an open, saturated set, and so its image $U_i = V_i/\mathbb{R}^\times \subseteq \mathbb{RP}^n$ is open. The $V_i$’s cover $\mathbb{R}^{n+1} \setminus \{0\}$, so the $U_i$’s cover $\mathbb{RP}^n$. We leave the rest of the details as an exercise.

5. $\mathbb{CP}^n$ is a $2n$-manifold. This is similar to the description given above.

6. $O(n)$ is a $\frac{n(n-1)}{2}$-manifold. Since it is also a topological group, this makes it a Lie group. The standard way to see that this is a manifold is to realize the orthogonal group as the preimage of the identity matrix under the transformation $M_n(\mathbb{R}) \to M_n(\mathbb{R})$ that sends $A$ to $A^T A$. This map lands in the subspace $S_n(\mathbb{R})$ of symmetric $n \times n$ matrices. This space can be identified with $\mathbb{R}^{n(n+1)/2}$.

   Now the $n \times n$ identity matrix is an element of $S_n$, and an important result in differential topology (Sard’s theorem) that says that if a certain derivative map is surjective, then the preimage of the submanifold $\{I_n\}$ will be a submanifold of $M_n(\mathbb{R})$ of the same “codimension”. In this case, the relevant derivative is the matrix of partial derivatives of $A \mapsto A^T A$, written in a suitable basis. It follows that

   $$\dim O(n) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

   The dimension statement can also be seen directly as follows. If $A$ is an orthogonal matrix, its first column is just a point of $S^{n-1}$. Then its second column is a point on the sphere orthogonal to the first column, so it lives in an “equator”, meaning a sphere of dimension one less. Continuing in this way, we see that the “degree of freedom” for specifying a point of $O(n)$ is $(n-1) + (n-2) + \cdots + 1 = \frac{n(n-1)}{2}$. 

7. $\text{Gr}_{k,n}(\mathbb{R})$ is a $k(n-k)$-manifold. One way to see this is to use the homeomorphism

   $$\text{Gr}_{k,n}(\mathbb{R}) \cong O(n)/(O(k) \times O(n-k))$$
from Example 18.1. We get
\[
\dim \text{Gr}_{n,k}(\mathbb{R}) = \dim O(n) - \left( \dim O(k) + \dim O(n-k) \right)
\]
\[
= \sum_{j=1}^{n-1} j - \left( \sum_{j=1}^{k-1} j + \sum_{\ell=1}^{n-k-1} \ell \right) = \sum_{j=k}^{n-1} j - \sum_{\ell=0}^{n-k-1} \ell
\]
\[
= \sum_{\ell=0}^{n-k-1} k + \ell - \sum_{\ell=0}^{n-k-1} \ell = \sum_{\ell=0}^{n-k-1} k = k(n-k)
\]

Here are some nonexamples of manifolds.

Example 36.3.  
(1) The union of the coordinate axes in \(\mathbb{R}^2\). Every point has a neighborhood like \(\mathbb{R}^1\) except for the origin.
(2) A discrete uncountable set is not second countable.
(3) A 0-manifold is discrete, so \(\mathbb{Q}\) is not a 0-manifold.
(4) Glue together two copies of \(\mathbb{R}\) by identifying any nonzero \(x\) in one copy with the point \(x\) in the other. This is second-countable and looks locally like \(\mathbb{R}^1\), but it is not Hausdorff.

37. Fri, Nov. 21

Proposition 37.1. Any manifold is normal.

Proof. This follows from Theorem 34.3. To see that a manifold \(M\) is locally compact, consider a point \(x \in M\). Then \(x\) has a Euclidean neighborhood \(x \in U \subseteq M\). \(U\) is homeomorphic to an open subset \(V\) of \(\mathbb{R}^n\), so we can find a compact neighborhood \(K\) of \(x\) in \(V\) (think of a closed ball in \(\mathbb{R}^n\)). Under the homeomorphism, \(K\) corresponds to a compact neighborhood of \(x\) in \(U\).

It also follows similarly that any manifold is metrizable, but we can do better. It is convenient to introduce the following term.

Recall that the support of a continuous function \(f : X \to \mathbb{R}\) is \(\text{supp}(f) = f^{-1}(\mathbb{R} \setminus \{0\})\).

Definition 37.2. Let \(\mathcal{U} = \{U_1, \ldots, U_n\}\) be a finite cover of \(X\). A partition of unity subordinate to \(\mathcal{U}\) is a collection \(\varphi_j \to [0,1]\) of continuous functions such that
(1) \(\text{supp}(\varphi_j) \subseteq U_j\)
(2) we have \(\sum_j \varphi_j = 1\).

Theorem 37.3. Let \(\mathcal{U} = \{U_1, \ldots, U_n\}\) be a finite covering of the normal space \(X\). Then there is a partition of unity subordinate to \(\mathcal{U}\).

Lemma 37.4. Let \(\mathcal{U} = \{U_1, \ldots, U_n\}\) be a finite covering of the normal space \(X\). Then there is a finite cover \(\mathcal{V} = \{V_1, \ldots, V_n\}\) such that \(V_i \subseteq \overline{V}_i \subseteq U_i\) for all \(i\).

Proof. We give only the argument in the case \(n = 2\). Let \(A = X \setminus U_2\). Then \(A \subseteq U_1\), so we can find an open \(V_1\) with \(A \subseteq V_1 \subseteq \overline{V}_1 \subseteq U_1\). Now \(\{V_1, U_2\}\) is an open cover of \(X\). In the same way, we replace \(U_2\) be a \(V_2\) with \(X \setminus V_1 \subseteq V_2 \subseteq \overline{V}_2 \subseteq U_2\).

Proof of Theorem 37.3. We use the lemma twice, to get finite covers \(\{V_1, \ldots, V_n\}\) and \(\{W_1, \ldots, W_n\}\) with
\[
W_i \subseteq \overline{W}_i \subseteq V_i \subseteq \overline{V}_i \subseteq U_i
\]
for all \(i\). For each \(i\) we have a Urysohn function \(g_i : X \to [0,1]\) with \(g_i \equiv 1\) on \(\overline{W}_i\) and vanishing outside of \(V_i\). Note that this implies that \(\text{supp}(g_i) \subseteq \overline{V}_i \subseteq U_i\). Since the \(W_i\) cover \(X\), it follows that
if we define $G = \sum_i g_i$, then $G(x) \geq 1$ for all $x$. Thus $\varphi_i = g_i / G$ is a continuous function taking values in $[0, 1]$, and we get

$$\sum_i \varphi_i = \sum_i \frac{g_i}{G} = \frac{\sum_i g_i}{\sum_i g_i} = 1.$$  

\[\square\]

**Theorem 37.5.** Any manifold $M^n$ admits an embedding into some Euclidean space $\mathbb{R}^N$.

**Proof.** The theorem is true as stated, but we only prove it in the case of a compact manifold. Note that in this case, since $M$ is compact and $\mathbb{R}^N$ is Hausdorff, it is enough to find a continuous injection of $M$ into some $\mathbb{R}^N$.

Since $M$ is a manifold, it has an open cover by sets that are homeomorphic to $\mathbb{R}^n$. Since it is compact, there is a finite subcover $\{U_1, \ldots, U_k\}$. By Theorem 37.3, there is a partition of unity $\{\varphi_1, \ldots, \varphi_k\}$ subordinate to this cover. For each $i$, let $f_i : U_i \xrightarrow{\cong} \mathbb{R}^n$ be a homeomorphism. We can then piece these together as follows: for each $i = 1, \ldots, k$, define $g_i : M \rightarrow \mathbb{R}^n$ by

$$g_i(x) = \begin{cases} f_i(x)\varphi_i(x) & x \in U_i \\ 0 & x \in X \setminus \text{supp}(\varphi_i) \end{cases}.$$  

Note that $g_i$ is continuous by the glueing lemma, since $\text{supp}(\varphi_i)$ is closed. Then the $k$ functions $g_i$ together give a continuous function $g : M \rightarrow \mathbb{R}^{nk}$. Unfortunately, this need not be injective, since if $f_i(x) = 0$ and $x$ does not lie in any other $U_j$, it follows that $g(x) = 0$. Since there can be more than one such $x$, we cannot conclude that $g$ is injective.

One way to fix this would be to stick on the functions $\varphi_i$, in order to separate out points lying in different $U_i$'s. Define $G = (g_1, \ldots, g_k, \varphi_1, \ldots, \varphi_k) : M \rightarrow \mathbb{R}^{nk+k}$. But now $G$ is injective, since if $G(x) = G(x')$ and we pick $i$ so that $\varphi_i(x) = \varphi_i(x') > 0$, then this means that $x$ and $x'$ both lie in $U_i$. But then $g_i(x) = g_i(x')$ so $f_i(x) = f_i(x')$. Since $f_i$ is a homeomorphism, it follows that $x = x'$.

\[\square\]

In fact, one can do better. Munkres shows (Cor. 50.8) that every compact $n$-manifold embeds inside $\mathbb{R}^{2n+1}$.  

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Last time, we discussed some of the nice properties of manifolds. Here is one more we did not get to.

**Proposition 38.1.** Any manifold is locally path-connected.

This follows immediately since a manifold is locally Euclidean.

Another related concept is that of paracompactness. This is especially important in the theory of manifolds and vector bundles. We make a couple of preliminary definitions first.

**Definition 38.2.** If $\mathcal{U}$ and $\mathcal{W}$ are collections of subsets of $X$, we say that $\mathcal{W}$ is a refinement of $\mathcal{U}$ if every $W \in \mathcal{W}$ is a subset of some $U \in \mathcal{U}$.

**Definition 38.3.** An open cover $\mathcal{U}$ of $X$ is said to be locally finite if every $x \in X$ has a neighborhood meeting only finitely many elements of the cover.

For example, the covering $\{ (n, n+2) \mid n \in \mathbb{Z} \}$ of $\mathbb{R}$ is locally finite.

**Definition 38.4.** A space $X$ is said to be paracompact if every open cover has a locally finite refinement.

From the definition, it is clear that compact implies paracompact. But this really is a generalization, as the next example shows.

**Proposition 38.5.** The space $\mathbb{R}$ is paracompact.

**Proof.** Let $\mathcal{U}$ be an open cover of $\mathbb{R}$. For each $n \geq 0$, let $A_n = [-n,n+1]$ and $W_n = [n - \frac{1}{2}, n + \frac{3}{2}]$. Then $A_n \subset W_n$, $A_n$ is compact and $W_n$ is open. (We take $W_0 = (\frac{3}{2}, \frac{3}{2})$.) Fix an $n$. For each $x \in A_n$, pick a $U_x \in \mathcal{U}$ with $x \in U_x$, and let $V_x = U_x \cap W_n$. The $V_x$’s give an open cover of $A_n$, and so there is a finite collection $\mathcal{V}_n$ of $V_x$’s that will cover $A_n$. Then $\mathcal{V} = \bigcup_n \mathcal{V}_n$ gives a locally finite refinement of $\mathcal{U}$. (Note that only $W_{n-1}$, $W_n$, and $W_{n+1}$ meet the subset $A_n$.)

This argument adapts easily to show that $\mathbb{R}^n$ is paracompact. In fact, something more general is true.

**Lemma 38.6.** Any open cover of a second countable space has a countable subcover.

**Proof.** Given a countable basis $\mathcal{B}$ and an open cover $\mathcal{U}$, we first replace the basis by the countable subset $\mathcal{B'}$ consisting of those basis elements that are entirely contained in some open set from the cover (this is a basis too, but we don’t need that). For each $B \in \mathcal{B'}$, pick some $U_B \in \mathcal{U}$ containing $B$, and let $\mathcal{U'} \subseteq \mathcal{U}$ be the (countable) collection of such $U_B$. It only remains to observe that $\mathcal{U'}$ is still a cover, because

$$\bigcup_{\mathcal{U'}} U_B \supset \bigcup_{\mathcal{B'}} B = X.$$  

**Proposition 38.7.** Every second countable, locally compact Hausdorff space is paracompact.

The proof strategy is the same. The assumptions give you a cover (basis) by precompact sets and thus a countable cover by precompact sets. You use this to manufacture a countable collection of compact sets $A_n$ and open sets $W_n$ that cover $X$ as above. The rest of the proof is the same.

Note that of the assumptions in the proposition, locally compact and Hausdorff are both local properties, whereas second countable is a global property. As we will see, paracompactness (and therefore the assumptions in this proposition) is enough to guarantee the existence of some nice functions on a space.
Corollary 38.8. Any manifold is paracompact.

Theorem 38.9 (Munkres, Theorem 41.4). If X is metric, then it is paracompact.

Next, we show that paracompact and Hausdorff implies normal. First, we need a lemma.

Lemma 38.10. If \( A \) is a locally finite collection of subsets of X, then
\[
\bigcup A = \bigcup \overline{A}.
\]

Proof. We have already shown before that the inclusion \( \supset \) holds generally. The other implication follows from the neighborhood criterion for the closure. Let \( x \in \bigcup A \). Then we can find a neighborhood \( U \) of \( x \) meeting only \( A_1, \ldots, A_n \). Then \( x \in \bigcup_{i=1}^n A_i \) since else there would be a neighborhood \( V \) of \( x \) missing the \( A_i \)'s. Then \( U \cap V \) would be a neighborhood missing \( \bigcup A \). By \( \bigcup_{i=1}^n A_i = \bigcup_{i=1}^n \overline{A_i} \), so we are done.

Theorem 38.11 (Lee, Theorem 4.81). If X is paracompact and Hausdorff, then it is normal.

Proof. We first use the Hausdorff assumption to show that X is regular. A similar argument can then be made, using regularity, to show normality.

Thus let A be closed and \( b \not\in A \). We wish to find disjoint open sets \( A \subseteq U \) and \( b \in V \). For every \( a \in A \), we can find disjoint open neighborhoods \( U_a \) of a and \( V_a \) of b. Then \( \{U_a\} \cup \{X \setminus A\} \) is an open cover, so there is a locally finite subcover \( V \). Take \( W \subseteq V \) to be the \( W \in V \) such that \( W \subseteq U_a \) for some \( a \). Then \( W \) is still locally finite.

We take \( U = \bigcup_{W \in V} W \) and \( V = X \setminus U \). We know \( b \in V \) since \( U = \bigcup W \), and \( b \not\in W \) since \( W \subseteq U_a \) and b has a neighborhood \( (V_a) \) disjoint from \( U_a \).

Definition 38.12. Let \( \mathcal{U} = \{U_a\} \) be a cover of X. A partition of unity subordinate to \( \mathcal{U} \) is a collection \( \varphi_a : X \to [0,1] \) of continuous functions such that
1. \( \text{supp}(\varphi_a) \subseteq U_a \)
2. the collection \( \text{supp}(\varphi_a) \) is locally finite
3. we have \( \sum \varphi_a = 1 \). Note that, when evaluated at some \( x \in X \), this sum is always finite

Theorem 38.13. Let X be paracompact Hausdorff, and let \( \mathcal{U} = \{U_a\} \) be an open cover. Then there exists a partition of unity subordinate to \( \mathcal{U} \).

Lemma 38.14 (Lee, 4.84). There exists a locally finite refinement \( \{V_a\} \) of \( \{U_a\} \) with \( \overline{V_a} \subseteq U_a \).

Proof of Theorem. We apply the lemma twice to get locally finite covers \( \{V_a\} \) and \( \{W_a\} \) with \( \overline{W_a} \subseteq V_a \subseteq \overline{V_a} \subseteq U_a \). For each \( a \), we use Urysohn’s lemma to get \( f_a : X \to [0,1] \) with \( f_a \equiv 1 \) on \( \overline{W_a} \) and \( \text{supp}(f_a) \subseteq \overline{V_a} \subseteq U_a \). Since \( \{V_a\} \) is locally finite, we can define \( f : X \to [0,1] \) by \( f = \sum f_a \). Locally around some \( x \in X \), the function \( f \) is a finite sum of \( f_a \)'s, and so is continuous. It only remains to normalize our \( f_a \)'s. Note that at any \( x \in X \), we can find an \( a \) for which \( x \in W_a \), and so \( f(x) \geq f_a(x) = 1 \). Thus it makes sense to define \( \varphi_a : X \to [0,1] \) by
\[
\varphi_a(x) = \frac{f_a(x)}{f(x)}.
\]
We have \( \text{supp}(\varphi_a) = \text{supp}(f_a) \), and so the \( \varphi_a \) give a partition of unity.