39. Mon, Dec. 1

The last main topic from the introductory part of the course on metric spaces is the idea of a function space. Given any two spaces A and Y, we will want to be able to define a topology on the set of continuous functions $A \longrightarrow Y$ in a sensible way. We already know one topology on Y^A, namely the product topology. But this does not use the topology on A at all.

Let's forget about topology for a second. Recall from the beginning of the course that a function $h : X \times A \longrightarrow Y$ between sets is equivalent to a function

$$\Psi(h) : X \longrightarrow Y^A.$$

Given h, the map $\Psi(h)$ is defined by $\big(\Psi(h)(x)\big)(a) = h(x, a)$. Conversely, given $\Psi(h)$, the function h can be recovered by the same formula.

Let's play the same game in topology. What we want to say is that a continuous map $h : X \times A \longrightarrow Y$ is the same as a continuous map $X \longrightarrow \mathrm{Map}(A, Y)$, for some appropriate *space* of maps $\mathrm{Map}(A, Y)$. Let's start by seeing why the product topology does *not* have this property.

We write $\mathcal{C}(X, Z)$ for the *set* of continuous maps $X \longrightarrow Z$. It is not difficult to check that the set-theoretic construction from above does give a function

$$\mathcal{C}(X \times A, Y) \longrightarrow \mathcal{C}(X, Y^A),$$

where for the moment Y^A denotes the set of continuous functions $A \longrightarrow Y$ given the product topology. But this function is not surjective.

Example 39.1. Take $A = [0, 1]$, $Y = \mathbb{R}$, and $X = Y^A = \mathbb{R}^{[0,1]}$. We can consider the identity map $\mathbb{R}^{[0,1]} \longrightarrow \mathbb{R}^{[0,1]}$. We would like this to correspond to a continuous map $\mathbb{R}^{[0,1]} \times [0, 1] \longrightarrow \mathbb{R}$. We see that, ignoring the topology, this function must be the evaluation function $ev : (g, x) \mapsto g(x)$. But this is not continuous.

To see this consider $ev^{-1}((0, 1))$. The point $(\mathrm{id}, 1/2)$ lies in this preimage, but we claim that no neighborhood of this point is contained in the preimage. In fact, we claim no basic neighborhood $U \times (a, b)$ lies in the preimage. For such a U must consist of functions that are close to $\mathrm{id} : [0, 1] \longrightarrow \mathbb{R}$ at *finitely many points* $c_1, \ldots, c_n$. So given any such U and any interval $(a, b) = (1/2 - \epsilon, 1/2 + \epsilon)$, pick any point $d \in (a, b)$ that is distinct from the c_i. It is simple to construct a continuous function $g : [0, 1] \longrightarrow \mathbb{R}$ such that (1) $g(c_i) = c_i$ for each i and (2) $g(d) = 2$. Then $(g, d) \in U \times (a, b)$ but $(g, d) \notin ev^{-1}((0, 1))$ since $ev(g, d) = g(d) = 2$.

The **compact-open** topology on the set $\mathcal{C}(A, Y)$ has a prebasis given by

$$S(K, U) = \{f : A \longrightarrow Y \mid f(K) \subseteq U\},$$

where K is compact and $U \subseteq Y$ is open. We write $\mathrm{Map}(A, Y)$ for the set $\mathcal{C}(A, Y)$ equipped with the compact-open topology.

Theorem 39.2. *Suppose that A is locally compact Hausdorff. Then a function $f : X \times A \longrightarrow Y$ is continuous if and only if the induced function $g = \Psi(f) : X \longrightarrow \mathrm{Map}(A, Y)$ is continuous.*

Proof. ($\Rightarrow$) This direction does not need that A is locally compact. Before we give the proof, we should note why $\Psi(f)(x) : A \longrightarrow Y$ is continuous. This map is the composite $A \xrightarrow{\iota_x} X \times A \xrightarrow{f} Y$ and therefore continuous.

We now wish to show that $g = \Psi(f)$ is continuous. Let $S(K, U)$ be a sub-basis element in $\mathrm{Map}(A, Y)$. We wish to show that $g^{-1}(S(K, U))$ is open in X. Let $g(x) = f(x, -) \in S(K, U)$. Since f is continuous, the preimage $f^{-1}(U) \subseteq X \times A$ is open. Furthermore, $\{x\} \times K \subseteq f^{-1}(U)$. We wish to use the Tube Lemma, so we restrict from $X \times A$ to $X \times K$. By the Tube Lemma, we can find a basic neighborhood V of x such that $V \times K \subseteq (X \times K) \cap f^{-1}(U)$. It follows that $g(V) \subseteq S(K, U)$, so that V is a neighborhood of x in $g^{-1}(S(K, U))$.

($\Leftarrow$) Suppose that g is continuous. Note that we can write f as the composition

$$X \times A \xrightarrow{g\times \mathrm{id}} \mathrm{Map}(A, Y) \times A \xrightarrow{ev} Y,$$

so it is enough to show that ev is continuous.

Lemma 39.3. *The map $ev : \mathrm{Map}(A, Y) \times A \longrightarrow Y$ is continuous if A is locally compact Hausdorff.*

Proof. Let $U \subseteq Y$ be open and take a point (f, a) in $ev^{-1}(U)$. This means that $f(a) \in U$. Since A is locally compact Hausdorff, we can find a compact neighborhood K of a contained in $f^{-1}(U)$ (this is open since f is continuous). It follows that $S(K, U)$ is a neighborhood of f in $\mathrm{Map}(A, Y)$, so that $S(K, U) \times K$ is a neighborhood of (f, a) in $ev^{-1}(U)$. ■

■

40. Wed, Dec. 3

Even better, we have

Theorem 40.1. *Let X and A be locally compact Hausdorff. Then the above maps give homeomorphisms*

$$\mathrm{Map}(X \times A, Y) \cong \mathrm{Map}(X, \mathrm{Map}(A, Y)).$$

It is fairly simple to construct a continuous map in either direction, using Theorem 39.2. You should convince yourself that the two maps produced are in fact inverse to each other.

In practice, it's a bit annoying to keep track of these extra hypotheses at all times, especially since not all constructions will preserve these properties. It turns out that there is a "convenient" category of spaces, where everything works nicely.

Definition 40.2. A space A is **compactly generated** if a subset $B \subseteq A$ is closed if and only if for every map $u : K \longrightarrow A$, where K is compact Hausdorff, then $u^{-1}(B) \subseteq K$ is closed.

We say that the topology of A is determined (or generated) by compact subsets. Examples of compactly generated spaces include locally compact spaces and first countable spaces.

Definition 40.3. A space X is **weak Hausdorff** if the image of every $u : K \longrightarrow X$ is closed in X.

There is a way to turn any space into a weak Hausdorff compactly generated space. In that land, <u>everything</u> works well! For the most part, whenever an algebraic topologist says "space", they really mean a compactly generated weak Hausdorff space. Next semester, we will always implicity be working with spaces that are CGWH.

Looking back to the initial discussion of metric spaces, there we introduced the uniform topology on a mapping space.

Theorem 40.4 (Munkres, 46.7 or Willard, 43.6)**.** *Let Y be a metric space. Then on the set $\mathcal{C}(A, Y)$ of continuous functions $A \longrightarrow Y$, the compact-open topology is intermediate between the uniform topology and the product topology. Furthermore, the compact-open topology agrees with the uniform topology if A is compact.*

The main point is to show (Munkres, Theorem 46.8) that the compact-open topology can be described by basis elements

$$B_K(f, \epsilon) = \{g : A \longrightarrow Y \mid \sup_K d(f(x), g(x)) < \epsilon\}.$$

To see that this satisfies the intersection property for a basis, suppose that

$$g \in B_{K_1}(f_1, \epsilon_1) \cap B_{K_2}(f_2, \epsilon_2)$$

Write $m_i = \sup_{K_i} d(f_i(x), g(x))$ and $\delta_i = \epsilon_i - m_i$. Then the triangle inequality gives

$$B_{K_i}(g, \delta_i) \subseteq B_{K_i}(f_i, \epsilon_i).$$

It follows that, setting $\delta = \min\{\delta_1, \delta_2\}$

$$g \in B_{K_1 \cup K_2}(g, \delta) \subseteq B_{K_1}(f_1, \epsilon_1) \cap B_{K_2}(f_2, \epsilon_2).$$

In the setting of metric spaces, the compact-open topology is known as the topology of compact convergence, as convergence of functions corresponds to (uniform) convergence on compact subsets.

One of the good properties of the uniform topology is that tif (f_n) is sequence of continuous functions and $f_n \to f$ in the uniform topology, then f is continuous. In other words, if we denote by $\mathcal{F}(X, Y)$ the set of *all* functions $X \longrightarrow Y$, then

$$\mathcal{C}(X, Y)_{\text{unif}} \subseteq \mathcal{F}(X.Y)_{\text{unif}}$$

is closed. This also happens in the compact-open topology.

Proposition 40.5. *Suppose that X is locally compact. Then*

$$\mathrm{Map}(X, Y) \subseteq \mathcal{F}(X, Y)_{\text{compact}-\text{open}}$$

is closed.

For fun, here is one of the first results towards the theory of C^*-algebras (pronounced C-star).

Theorem 40.6. *Let X be compact Hausdorff and denote by $C(X)$ the space* $\mathrm{Map}(X, \mathbb{R})$ *of real-valued functions on X. Then the map*

$$\Lambda : X \longrightarrow \widehat{C(X)} = \{\lambda : C(X) \longrightarrow \mathbb{R} \mid \lambda \textit{ is a continuous } \mathbb{R}\textit{-algebra map}\}$$

defined by

$$\Lambda(x) = \mathrm{ev}_x$$

is a homeomorphism if $\widehat{C(X)} \subseteq \prod_{C(X)} \mathbb{R}$ is equipped with the product topology.

In this case, the product topology coincides with a topology of interest in analysis known as the weak-$*$ topology.

Proof. Since we have given $\widehat{C(X)}$ the product topology, it is simple to verify that Λ is continuous. Note that since X is compact and $\widehat{C(X)}$ is Hausdorff, it remains only to show that Λ is a bijection.

Suppose that $\Lambda(x) = \Lambda(x')$. Since X is compact Hausdorff, for any two distinct points there is a continuous function taking different values at those points. The fact that $\Lambda(x) = \Lambda(x')$ says that no such function exists for x and x', so we must have $x = x'$.

Now let $\lambda \in \widehat{C(X)}$. We wish to show that $\lambda = \mathrm{ev}_x$ for some x.

The main step is to show that there exists $x \in X$ such that if $\lambda(f) = 0$ for some f, then $f(x) = 0$. Suppose not. Then for every $x \in X$, there exists a function f_x with $\lambda(f_x) = 0$ but $f_x(x) \neq 0$. For each x, let $U_x = f_x^{-1}(\mathbb{R} \setminus \{0\})$. Then the collection $\{U_x\}$ covers X since $x \in U_x$. As X is compact, there is a finite subcover $\{U_{x_1}, \ldots, U_{x_n}\}$. Now define

$$g = f_{x_1}^2 + \cdots + f_{x_n}^2,$$

and note that $g(x) > 0$ for all x. This is because $f_{x_i} \neq 0$ on U_{x_i} and the U_{x_i} cover X. Since g is nonzero, it follows that $1/g$ is also continuous on X. But now

$$1 = \lambda(g \cdot 1/g) = \lambda(g) \cdot \lambda(1/g),$$

which implies that $\lambda(g) \neq 0$. But λ is an algebra homomorphism, so

$$\lambda(g) = \sum_i \lambda(f_{x_i})^2 > 0,$$

which is a contradiction.

This now establishes that there must be an $x \in X$ such that if $\lambda(f) = 0$ then $f(x) = 0$. But now the theorem follows, for if $f \in C(X)$, then

$$\lambda(f - \lambda(f) \cdot 1) = \lambda(f) - \lambda(f) \cdot \lambda(1) = 0.$$

By the above, we then have that $f(x) - \lambda(f) = 0$, so that $\lambda(f) = f(x)$. In other words, $\lambda = \mathrm{ev}_x$. ■

Recently, we consider topological manifolds, which are a nice collection of spaces. Next semester, you will often work with another nice collection of spaces that can be built inductively. These are cell complexes, or CW complexes.

A typical example is a sphere. In dimension 1, we have S^1, which we can represent as the quotient of $I = [0, 1]$ by endpoint identification. Another way to say this is that we start with a point, and we "attach" an interval to that point by gluing both ends to the given point.

For S^2, there are several possibilities. One is to start with a point and glue a disk to the point (glueing the boundary to the point). An alternative is to start with a point, then attach an interval to get a circle. To this circle, we can attach a disk, but this just gives us a disk again, which we think of as a hemisphere. If we then attach a second disk (the other hemisphere), we get S^2.

But what do we really mean by "attach a disk"?

Let's start today by discussing the general "pushout" construction.

Definition 40.7. Suppose that $f : A \longrightarrow X$ and $g : A \longrightarrow Y$ are continuous maps. The **pushout** (or glueing construction) of X and Y along A is defined as

$$X \cup_A Y := X \amalg Y / \sim, \qquad f(a) \sim g(a).$$

We have an inclusion $X \hookrightarrow X \amalg Y$. Composing this with the quotient map to $X \cup_A Y$ gives the map $\iota_X : X \longrightarrow X \cup_A Y$. We similarly have a map $\iota_Y : Y \longrightarrow X \cup_A Y$. Moreover, these maps make the diagram to the right commute. The point is that

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\iota_Y"] \\ X \arrow[r, "\iota_X"] & X \cup_A Y \end{tikzcd}$$

$$\iota_X(f(a)) = \overline{f(a)} = \overline{g(a)} = \iota_Y(g(a)).$$

The main point of this construction is the following property.

Proposition 40.8 (Universal property of the pushout)**.** *Suppose that $\varphi_1 : X \longrightarrow Z$ and $\varphi_2 : Y \longrightarrow Z$ are maps such that $\varphi_1 \circ f = \varphi_2 \circ g$. Then there is a unique map $\Phi : X \cup_A Y \longrightarrow Z$ which makes the diagram to the right commute.*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\iota_Y"] \arrow[ddr, bend left, "\varphi_2"] & \\ X \arrow[r, "\iota_X"] \arrow[drr, bend right, "\varphi_1"'] & X \cup_A Y \arrow[dr, dashed, "\Phi"] & \\ & & Z \end{tikzcd}$$

This generalizes the "pasting" lemma. Suppose that $U, V \subseteq X$ are open subsets. Then it is not difficult to show that the pushout $U \cup_{U \cap V} V$ is homeomorphic to X. The universal property for the pushout then says that specifying a continuous map out of X is the same as specifying a pair of continuous maps out of U and V which agree on their intersection $U \cap V$. This is precisely the statement of the pasting lemma!

Definition 40.9. (Attaching an interval) Given a space X and two points $x \neq y \in X$, we get a continuous map $\alpha : S^0 \longrightarrow X$ with $\alpha(0) = x$ and $\alpha(1) = y$. There is the standard inclusion $S^0 \hookrightarrow D^1 = [-1, 1]$, and we write $X \cup_\alpha D^1$ for the pushout

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d, "\alpha"'] & D^1 \arrow[d] \\ X \arrow[r, "\iota_X"] & X \cup_\alpha D^1 \end{tikzcd}$$

The image $\iota(\mathrm{Int}(D^1))$ is referred to as a 1-cell and is sometimes denoted e^1. Thus the above space, which is described as obtained by attaching an 1-cell to X, is also written $X \cup_\alpha \overline{e^1}$ or $X \cup_\alpha e^1$.

41. Fri, Dec. 5

Generalizing the construction from last time, for any n, we have the standard inclusion $S^{n-1} \hookrightarrow D^n$ as the boundary.

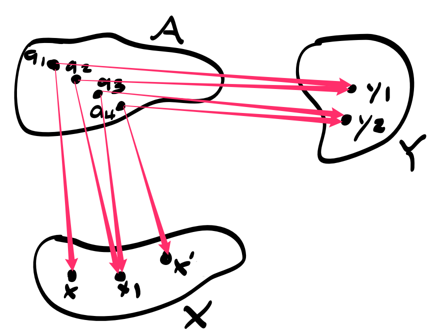
Definition 41.1. Given a space X and a continuous map $\alpha : S^{n-1} \longrightarrow X$, we write $X \cup_\alpha D^n$ for the pushout

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, "\alpha"'] & D^n \arrow[d] \\ X \arrow[r, "\iota_X"] & X \cup_\alpha D^n \end{tikzcd}$$

The image $\iota(\mathrm{Int}(D^n))$ is referred to as an n-cell and is sometimes denoted e^n. Thus the above space, which is described as obtained by attaching an n-cell to X, is also written $X \cup_\alpha \overline{e^n}$ or $X \cup_\alpha e^n$.

In general, this attaching process does not disturb the interiors of the cells, as follows from

Proposition 41.2. *If $g : A \hookrightarrow Y$ is injective, then $\iota_X : X \longrightarrow X \cup_A Y$ is also injective.*



Proof. Suppose that $\iota_X(x) = \iota_X(x')$. The relation imposed on $X \amalg Y$ only affects points in $f(A)$ and $g(A)$. We assume that $x, x' \in f(A)$ since otherwise we must have $x = x'$. In general, the situation we should expect is represented in the picture to the right. But since g is injective, this means that $a_1 = a_2$ and $a_3 = a_4$. This implies that $x = f(a_1) = f(a_2) = x_1$ and that $x_1 = f(a_3) = f(a_4) = x'$. Putting these together gives $x = x'$. ■

Example 41.3. If $A = \emptyset$, then $X \cup_A Y = X \amalg Y$.

Example 41.4. If $A = *$, then $X \cup_A Y = X \vee Y$.

Example 41.5. If $A \subseteq X$ is a subspace and $Y = *$, then $X \cup_A * \cong X/A$.

By the way, Proposition 41.2 is not only true for injections.

Proposition 41.6. *(i) If $f : A \longrightarrow X$ is surjective, then so is $\iota_Y : Y \longrightarrow X \cup_A Y$.*

(ii) If $f : A \longrightarrow X$ is a homeomorphism, then so is $\iota_Y : Y \longrightarrow X \cup_A Y$.

Proof. We prove only (ii). We show that if f is a homeomorphism, then Y satisfies the same universal property as the pushout. Consider the test diagram to the right. We have no choice but to set $\Phi = \varphi_2$. Does this make the diagram commute? We need to check that $\Phi \circ g \circ f^{-1} = \varphi_1$. Well,

$$\Phi \circ g \circ f^{-1} = \varphi_2 \circ g \circ f^{-1} = \varphi_1 \circ f \circ f^{-1} = \varphi_1.$$

$$\begin{tikzcd}
A \arrow[rr, "g"] \arrow[d, "f"'] & & Y \arrow[d, equal] \arrow[ddr, bend left, "\varphi_2"] & \\
X \arrow[r, "f^{-1}"] \arrow[drrr, bend right, "\varphi_1"'] & A \arrow[r, "g"] & Y \arrow[dr, dashed, "\Phi"] & \\
 & & & Z
\end{tikzcd}$$

■

We use the idea of attaching cells (using a pushout) to inductively build up the idea of a cell complex or CW complex.

Definition 41.7. A **CW complex** is a space built in the following way

(1) Start with a discrete set X^0 (called the set of 0-cells, or the 0-skeleton)
(2) Given the $(n-1)$-skeleton X^{n-1}, the n-skeleton X^n is obtained by attaching n-cells to X^{n-1}.
(3) The space X is the union of the X^n, topologized using the "weak topology". This means that $U \subseteq X$ is open if and only if $U \cap X^n$ is open for all n.

The third condition is not needed if $X = X^n$ for some n (so that X has no cells in higher dimensions). On the other hand, the 'W' in the name CW complex refers to item 3 ("weak topology"). The 'C' in CW complex refers to the **C**losure finite property: the closure of any cell is contained in a finite union of cells. We will come back to this point later.

According to condition (2), the n-skeleton is obtained from the $(n-1)$-skeleton by attaching cells. Often, we think of this as attaching one cell at a time, but we can equally well attach them all at once, yielding a pushout diagram

$$\begin{tikzcd}
\coprod_{\mathcal{E}_n} S^{n-1} \arrow[r] \arrow[d] & \coprod_{\mathcal{E}_n} D^n \arrow[d] \\
X^{n-1} \arrow[r] & X^n
\end{tikzcd}$$

for each n. The maps $S^{n-1} \longrightarrow X^{n-1}$ are called the **attaching maps** for the cells, and the resulting maps $D^n \longrightarrow X^n$ are called the **characteristic maps**.

Example 41.8. (1) S^n. We have already discussed two CW structures on S^2. The first has X^0 a singleton and a single n-cell attached. The other had a single 0-cell and single 1-cell but two 2-cells attached. There is a third option, which is to start with two 0-cells, attach two 1-cells to get a circle, and then attach two 2-cells to get S^2.

The first and third CW structures generalize to any S^n. There is a minimal CW structure having a single 0-cell and single n-cell, and there is another CW structure have two cells in every dimension up to n.