17. Mon, Oct. 6

(5) Similarly, we can think of $\mathbb{Z}^n$ acting on $\mathbb{R}^n$, and the quotient is $\mathbb{R}^n/\mathbb{Z}^n \cong (S^1)^n = T^n$.

(6) The group $Gl(n)$ acts on $\mathbb{R}^n$ (just multiply a matrix with a vector), but this is not terribly interesting, as there are only two orbits: the origin is a closed orbit, and the complement is an open orbit. Thus the quotient space consists of a closed point and an open point.

(7) More interesting is the action of the subgroup $O(n)$ on $\mathbb{R}^n$. Using the fact that orthogonal matrices preserve norms, it is not difficult to see that the orbits are precisely the spheres around the origin. We claim that the quotient is the space $[0, \infty)$ (thought of as a subspace of $\mathbb{R}$).

To see this, consider the continuous surjection $|-| : \mathbb{R}^n \longrightarrow [0, \infty)$. By considering how this acts on open balls, you can show that this is an open map and therefore a quotient. But the fibers of this map are precisely the spheres, so it follows that this is the quotient induced by the above action of $O(n)$.

At the end of class last time, we were looking at the example of $O(n)$ acting on $\mathbb{R}^n$, and we claimed that the quotient was $[0, \infty)$. We saw that the relation coming from the $O(n)$-action was the same as that coming from the surjection $\mathbb{R}^n \longrightarrow [0, \infty)$. Namely, we identify points if and only if they have the same norm. To see that the quotient by the $O(n)$-action is homeomorphic to $[0, \infty)$, it remains to show that the norm map $\mathbb{R}^n \longrightarrow [0, \infty)$ is a quotient map. We know already that it is a continuous surjection, and by considering basis elements (open balls) in $\mathbb{R}^n$, we can see that it is open as well. We leave this verification to the industrious student!

Why does the above argument show that the quotient $\mathbb{R}^n/O(n)$ is homeomorphic to $[0, \infty)$. We now have two quotient maps out of $\mathbb{R}^n$, and they are defined using the same equivalence relation on $\mathbb{R}^n$. By the universal property of quotients, the two spaces are homeomorphic!

Let's get on with more examples.

Example 17.1. (1) Let $\mathbb{R}^\times$ act on $\mathbb{R}^n$ via scalar multiplication. This action preserves lines, and within each line there are two orbits, one of which is the origin. Note that the only saturated open set containing 0 is $\mathbb{R}^n$, so the only neighborhood of 0 in the quotient is the entire space.

(2) Switching from n to $n+1$ for convenience, we can remove that troublesome 0 and let $\mathbb{R}^\times$ act on $X_{n+1} = \mathbb{R}^{n+1} \setminus \{0\}$. Here the orbits are precisely the lines (with origin removed). The quotient is $\mathbb{RP}^n$.

To see this, recall that we defined $\mathbb{RP}^n$ as the quotient of S^n by the relation $\mathbf{x} \sim -\mathbf{x}$. This is precisely the relation that arises from the action of the subgroup $C_2 = \{1, -1\} \leq \mathbb{R}^\times$ on $S^n \subseteq \mathbb{R}^{n+1}$.

Now notice that the map $\mathbb{R}^{n+1} \setminus \{0\} \longrightarrow S^n \times \mathbb{R}_{>0}$ given by $\mathbf{x} \mapsto \left(\frac{\mathbf{x}}{\|\mathbf{x}\|}, \|\mathbf{x}\|\right)$ is a homeomorphism. Next, note that we have an isomorphism $\mathbb{R}^\times \cong C_2 \times \mathbb{R}^\times_{>0}$. Thus the quotient $(\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^\times$ can be viewed as the two step quotient $\Big((S^{n-1} \times \mathbb{R}_{>0})/\mathbb{R}^\times_{>0})\Big)/C_2$. But $(\mathbb{R}^{n-1} \times \mathbb{R}_{>0})/\mathbb{R}^\times_{>0} \cong S^{n-1}$, so we are done.

We can think of $\mathbb{RP}^n$ in yet another way. Consider the following diagram:

$$\begin{array}{ccccc} D^n & \longrightarrow & S^n & \longrightarrow & \mathbb{R}^{n+1} \setminus \{0\} \\ \downarrow & & \downarrow & & \downarrow \\ D^n/\sim & \longrightarrow & S^n/C_2 & \longrightarrow & \mathbb{R}^{n+1} \setminus \{0\}/\mathbb{R}^\times \end{array}$$

The map $D^n \longrightarrow S^n$ is the inclusion of a hemisphere. The relation on D^n is the relation $\mathbf{x} \sim -\mathbf{x}$, but only allowed *on the boundary* ∂D^n. All maps on the bottom are continuous bijections, and again we will see later that they are necessarily homeomorphisms.

Note that the relation we imposed on D^n does *not* come from an action of C_2 on D^n. Let us write $C_2 = \langle \sigma \rangle$. We can try defining

$$\sigma \cdot \mathbf{x} = \begin{cases} \mathbf{x} & \mathbf{x} \in \mathrm{Int}(D^n) \\ -\mathbf{x} & \mathbf{x} \in \partial(D^n), \end{cases}$$

where here the interior and boundary are taken in S^n. But this is not continuous, as the convergent sequence

$$\left(\sqrt{1-\frac{1}{n}}, 0, \dots, 0, \sqrt{\frac{1}{n}}\right) \to (1, 0, \dots, 0)$$

is taken by σ to a convergent sequence, but the new limit is not $\sigma(1, 0, \dots, 0) = (-1, 0, \dots, 0)$.

18. Wed, Oct. 8

(3) We have a similar story for $\mathbb{CP}^n$. There is an action of $\mathbb{C}^\times$ on $\mathbb{C}^{n+1} \setminus \{0\}$, and the orbits are the punctured complex lines. We claim that the quotient is $\mathbb{CP}^n$.

We defined $\mathbb{CP}^n$ as a quotient of an S^1-action on S^{2n+1}. We also have a homeomorphism $\mathbb{C}^{n+1} \setminus \{0\} \cong S^{2n+1} \times \mathbb{R}_{>0}$ and an isomorphism $\mathbb{C}^\times \cong S^1 \times \mathbb{R}^\times_{>0}$. We can then describe $\mathbb{CP}^n$ as the two-step quotient

$$\left(\mathbb{C}^{n+1} \setminus \{0\}\right)/\mathbb{C}^\times \cong \left((S^{2n+1} \times \mathbb{R}_{>0})/\mathbb{R}^\times_{>0}\right)/S^1 \cong S^{2n+1}/S^1 = \mathbb{CP}^n.$$

We have been studying actions of topological groups on spaces, and the resulting quotient spaces X/G. But there is another way to think about this material. Suppose you have a set Y that you would like to topologize. One way to create a topology on Y is as follows. Pick a point $y_0 \in Y$. If there is a transitive action of some topological group G on Y, then the orbit-stabilizer theorem asserts that Y can be identified with G/H, where $H \leq G$ is the stabilizer subgroup consisting of all $h \in G$ such that $h \cdot y_0 = y_0$. But G/H is a topological space, so we define the topology on Y to be the one coming from the bijection $Y \cong G/H$.

Example 18.1. (Grasmannian) We saw that the projective spaces can be identified with the set of lines in $\mathbb{R}^n$ or $\mathbb{C}^n$, suitably topologized. We can similarly consider the set of k-dimensional linear subspaces in $\mathbb{R}^n$ (or $\mathbb{C}^n$). It is not clear how to topologize this set.

However, there is a natural action of $O(n)$ on the set of k-planes in $\mathbb{R}^n$. Namely, if $A \in O(n)$ is an orthogonal matrix and $V \subseteq \mathbb{R}^n$ is a k-dimensional subspace, then $A(V) \subseteq \mathbb{R}^n$ is another k-dimensional subspace. Furthermore, this action is transitive. To see this, it suffices to show that given any subspace V, there is a matrix taking the standard subspace $E_k = \mathrm{Span}\{\beta_1, \dots, \beta_k\}$ to V. Thus suppose $V = \mathrm{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a k-dimensional subspace with given orthogonal basis. This can be completed to an orthogonal basis of $\mathbb{R}^n$. Then if A is the orthogonal matrix with columns the $\mathbf{v}_i$, A takes the standard subspace E_k to V.

The stabilizer of E_k is the subgroup of orthogonal matrices that take E_k to E_k. Such matrices are block matrices, with an orthogonal $k \times k$ matrix in the upper left and an orthogonal $(n-k) \times (n-k)$ matrix in the lower right. In other words, the stabilizer subgroup is $O(k) \times O(n-k)$. It follows that the set of k-planes in $\mathbb{R}^n$ can be identified with the quotient

$$\mathrm{Gr}_{k,n}(\mathbb{R}) = O(n)/O(k) \times O(n-k).$$

Note that, from this identification, it is clear that $\mathrm{Gr}_{k,n} \cong \mathrm{Gr}_{n-k,n}$. The map takes a k-plane in $\mathbb{R}^n$ to the orthogonal complement, which is an $n-k$-plane in $\mathbb{R}^n$. The corresponding map

$$O(n)/O(k) \times O(n-k) \longrightarrow O(n)/O(n-k) \times O(k)$$

is induced by a map $O(n) \longrightarrow O(n)$. This map on $O(n)$ is conjugation by a shuffle permutation that permutes k things past $n-k$ things.

There is an identical story for the complex Grasmannians, where $O(n)$ is replaced by $U(n)$.

Example 18.2. (Flag varieties) Continuing (why not?) in this vein, we can consider the sets of flags in $\mathbb{R}^n$ or $\mathbb{C}^n$. Recall that a flag is a chain of strict inclusions of linear subspaces $0 \leq V_1 \leq V_2 \leq \cdots \leq V_k = \mathbb{R}^n$. A flag is said to be **complete** if $\dim V_k = k$. The general linear group $\mathrm{Gl}_n(\mathbb{R})$ acts transitively on the set of complete flags. Indeed, there is the standard complete flag $0 \leq E_1 \leq E_2 \leq \dots$, where $E_k = \mathrm{Span}\{\beta_1, \dots, \beta_k\}$, as above. Let $0 \leq V_1 \leq V_2 \leq \dots$ be any other complete flag. Then if we choose a basis $\{\mathbf{v}_i\}$ for $\mathbb{R}^n$ such that $V_k = \mathrm{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$, then it follows that the matrix A having the $\mathbf{v}_i$ for columns will take E_k to V_k.

In order to obtain a description of the complete flag variety $F(\mathbb{R}^n)$ as a space, we need to identify the stabilizer subgroup of a point. Let's look at the stabilizer of the standard complete flag. We want to know which matrices A will satisfy $A(\beta_k) \in E_k$ for all k. The vector $A(\beta_1)$ must be a nonzero multiple of β_1, so the only nonzero entry in the first column of A is the top left entry. A similar analysis of the other columns shows that A must be upper-triangular (and nonsingular). If we thus denote the subgroup of upper-triangular matrices by B_n (B is for 'Borel'), then we see that the flag variety can be identified with the topological space

$$F(\mathbb{R}^n) \cong \mathrm{Gl_n}(\mathbb{R})/B_n.$$

For some purposes, it is more convenient to work with the orthogonal group rather than the general linear group. This presents no real difficulty. We work with an orthonormal basis rather than any basis. Here we can see that the stabilizer of the standard flag consists of upper triangular orthogonal matrices, which coincides with the group of diagonal orthogonal matrices. These can only have 1 or -1 on the diagonal. We conclude that

$$F(\mathbb{R}^n) \cong O(n)/C_2 \times C_2 \times \cdots \times C_2.$$

There is a complex analogue as well. We have

$$F(\mathbb{C}^n) \cong \mathrm{Gl_n}(\mathbb{C})/B_n(\mathbb{C}) \cong U(n)/S^1 \times \cdots \times S^1 = U(n)/T^n.$$

What about non-complete flags? It is clear that if $\{V_i\}$ is a flag, then $\{AV_i\}$ will have the same "signature" (sequence of dimensions). But similar arguments to those above show that the general linear group or orthogonal group act transitively on the set of flags of a given signature, and we have

$$F(d_1, \dots, d_k; \mathbb{R}^n) \cong \mathrm{Gl_n}(\mathbb{R})/B_{n_1,\dots,n_k} \cong O(n)/O(n_1) \times O(n_2) \times \cdots \times O(n_k),$$

where $n_i = d_i - d_{i-1}$ and $B_{n_1,\dots,n_k}$ is the set of block-upper triangular matrices (with blocks of size n_1, n_2, etc.). Similarly,

$$F(d_1, \dots, d_k; \mathbb{C}^n) \cong \mathrm{Gl_n}(\mathbb{C})/B_{n_1,\dots,n_k}(\mathbb{C}) \cong U(n)/U(n_1) \times \cdots \times U(n_k).$$

19. Fri, Oct. 10

Exam day!